HINTS AND ANSWERS;

BEING A

KEY

TO A COLLECTION OF

CAMBRIDGE MATHEMATICAL

EXAMINATION PAPERS,

AS PROPOSED AT

THE SEVERAL COLLEGES.

BY

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AUTHOR OF THE PRIVATE TUTOR, A TRANSLATION OF NEWTON'S PRINCIPIA, SECTIONS I. II. III., &c. &c. &c.

PART I.

CONTAINING EUCLID, ARITHMETIC, AND ALGEBRA.

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PREFACE.

This work comprises not only *Hints* for the solution of the more difficult parts of questions, but *Answers* for all, over and above the complete developement of many of the most intricate. The full proof, or investigation, is bestowed upon nearly all the last five Algebraic Problems and Equations in each of those papers given at St. John's College which invariably consists of seven questions,—these papers having been found far more difficult of treatment than any proposed at other Colleges. Interesting questions also have been somewhat fully solved and exemplified; and the shortest practical methods have constantly been suggested on every available occasion. Notwithstanding so much has been done for the student, the work does not greatly exceed double the bulk of the questions themselves.

It were almost superfluous to direct students in the proper use of these aids. Let no Hint, Answer, or Solution be referred to, before the student's own patient efforts have entirely failed. A reference then may be equally advantageous with the

help of a tutor. But less benefit will accrue in proportion to the contrary being practised. To boys at school, whose only aim is to get through their tasks, Keys are detrimental; but to students at an University, where high honours and rewards await distinguished success at Examinations, the motives being essentially different, they will be rendered to good account—at least, by those who value their own progress in knowledge.

As the subjects of Geometry or Euclid, Arithmetic and Algebra, form a principal part of the studies for the degree of B.A., as well as for the Examination of Freshmen, it is presumed the volume will not prove unwelcome to those Undergraduates, who are considered the "*non-reading*" portion of the University.

Another volume, containing "Hints and Answers in *Trigonometry, and the Differential Calculus,*" is in progress, and will speedily be published.

Trinity House, Christ's Piece, Cambridge.
March 1, 1831.

BOOKS REFERRED TO.

Euclid's Elements. *Simson's.*
Wright's Self-Examinations in Euclid.
Creswell's Maxima and Minima.
Bland's Geometrical Problems.
Wright's Pure Arithmetic.
Wood's Algebra.
Wright's Private Tutor.
Barlow's Theory of Numbers.

HINTS AND ANSWERS

IN

EUCLID, ARITHMETIC, & ALGEBRA.

EUCLID.

TRINITY COLLEGE, 1824.

[P. 1.

1. The principles of construction are the three Postulates of Euclid.

2. Euclid, Book i. prop. 4.

3. Take any right-angled $\triangle$ ABC, right-angled at C; from C draw CM $\perp$ AB, thus obtaining three right-angled $\triangle^s$. Then, since the $\triangle^s$ ACM, ACB, have two angles in each equal, their third angles are equal; that is, $\angle$ ACM $=$ $\angle$ B. Similarly, it may be shown that $\angle$ BCM $=$ $\angle$ A.

Hence, $\angle \text{A} + \angle \text{B} = \angle \text{ACM} + \angle \text{BCM} =$ a right angle. *Consequently the two angles of a right-angled $\triangle$ which are not the right angle,* are equal to one right angle.

Again, take any $\triangle$ whatever abc, and from c draw $cm \perp ab$; then, by what has been shown,

$$\angle a + \angle acm = \text{one right } \angle,$$
$$\text{and } \angle b + \angle bcm = \text{one right } \angle,$$
$$\therefore \angle a + \angle b + \angle c = \text{two right } \angle^s.$$

4. Euclid, i. 35.

5. Let P be the given point in the side AB of the given $\triangle$ ABC. Suppose the thing done, and that the required line, bisecting the $\triangle$, meets the side BC in the point Q; then the $\triangle$ BPQ $= \frac{1}{2}$ $\triangle$ ABC.

Complete the parallelogram BPQR. This parallelogram $= 2 \triangle$ BPQ $= \triangle$ ABC. Whence the Synthetic construction and demonstration are evident.

For upon BP describe any parallelogram equal to the $\triangle$ ABC, (Euclid, i. 44), and let Q be the point where the upper side of the parallelogram enters the side BC. Join PQ. Then $\because$ &c. as is evident.

6. Euclid, i. 46, and *Wright's* Self-Exam. in Euclid, p. 4.

7. Euclid, i. 47. 8. Euclid, ii. 14. 9. Euclid, iii. 2.

10. *Wood,* Alg. art. 515. 11. Euclid, iv. 11.

12. They are an equilateral $\triangle$, a square, and a regular hexagon.

13. For problems of this kind, see *Creswell's* Maxima and Minima.

14. See *Wright's* Self-Exam. in Euclid, pp. 73, 74, 75, 76.

15. Euc. v. 12. 16. Euc. vi. 1. 17. Euc. vi. 4. 18. Euc. vi. 25.

19. To the side C of the rectangle $A \times C$ apply a rectangle $= B \times D$; then the base of $A \times C$ has to the base of the whole rectangle the ratio required.

20. For if the straight lines AB, CD cut the given lines AC, BD in A, B; C, D, respectively, and intersect in P; and likewise two other straight lines A′B′, C′D′, *parallel* to the former, cut the same given lines in A′, B′, and C′, D′, respectively, and intersect in P′; then the $\triangle^s$ APC, A′P′C′ are similar, and also DPB, and D′P′B′. $\therefore$ &c.

21. There are not data sufficient that the square shall be constructed *in position,* but only in magnitude; for it is clear the square, when constructed, from these given distances, might all assume any position around the point.

If a, b, be the two shorter distances, and c the longest; then it may easily be shown that the side of the square is

$$\sqrt{\frac{b^2+c^2 \pm \sqrt{\{4b^2c^2-(2a^2-b^2-c^2)^2\}}}{2}},$$

whence the construction is easy.

One solution belongs to that case in which the given point lies without the square; the other to that in which it lies within the square.

The problem is impossible when

$$2a^2-b^2-c^2 \text{ is } > 2bc,$$

$$\text{or when } a \text{ is } > \frac{b+c}{\sqrt{2}},$$

or when $\frac{a}{b+c}$ is $> \frac{1}{\sqrt{2}} >$ ratio of the side of a square to its diagonal.

This is also evident, from the consideration that two sides of a $\triangle$ must be always greater than the third side.

22. Join the two given points A, B; bisect AB by the perpendicular ED, meeting the given line CD in D. Then the centre of the required circle is in ED; and the distance of its centre O from D is

$$\frac{DE^3 \pm DE\sqrt{(DE^2.\ EF^2+EF^2.\ EB^2-ED^2.\ EF^2)}}{DE^2-EF^2},$$

EF making an $\angle$ with CD equal the given $\angle$.

23. Let AB be any side of the regular polygon, C being the centre of the circumscribed circle. Bisect the $\angle$ A′CB by the straight line CM; then AM = MB. Again, bisect the $\angle$ MCB by Cn, and draw Cm', Cb' $\perp$ Cn, and make each of them $= \frac{1}{2}$ MB. Draw $m'm$, $b'b$ parallel to Cn, to meet CM, CB in m, b; join mb; then, it is easily shown, that $mb=$ MB. In like manner draw $ma =$ AM; am, mb, will be two sides of the new regular polygon; and, similarly, all the others may be found.

TRINITY COLLEGE, 1828.

1. Euc. i. 24. 2. Euc. i. 45. 3. Euc. ii. 7. 4. Euc. ii. 11. 5. Euc. iii. 27. 6. Euc. iii. 33. 7. Euc. iv. 5. 8. Euc. iv. 13. 9. Euc. v. 10. 10. Euc. v. 22. 11. Euc. vi. 22. 12. Euc. vi. 33.

13. Let the straight lines AC, BD, cut by AB, make the angles BAC, ABD together less than two right angles. Draw

AE, making the angle BAE + angle ABD equal two right angles; then, by the assumption, AE and BD cannot meet. Consequently, AC and BD will meet.

14. This is easily proved, in a manner similar to that of Prop. 47. B. i. of Euclid.

15. First, make a square = given rectangle (Euclid, ii. 14). Next, on the given line describe a semi-circle; at the extremity of the line draw a $\perp$ which is = the side of the square; from the end of this $\perp$ draw a line $\perp$ to it meeting the semicircle in a point from which the $\perp$ upon the given line will divide it as required.

16. See *Bland's* Geomet. Problems, p. 231; or make use of Euclid, iv. 10.

17. This depends upon Euclid, ii. 12 and 13, and upon B. vi. prop. B.

18. Let the chords AB, *ab*, be produced to C; and let O be the centre of the circle; join OA, OB, O*a*, O*b*, OC, and produce CO to the circumference in D.

$$\text{Then } \angle DOA = \angle OAC + \angle OCA = \angle OBA + \angle OCA,$$
$$= \angle BOC + 2\,\angle OCA,$$
$$\therefore \frac{\angle DOA - \angle BOC}{2} = \angle OCA;$$
&c.

19. From one extremity A of the base of AB of the $\triangle$ as a centre and a radius = sum of the undetermined sides describe an arc. From the other extremity B draw any line BC to the circumference; join AC, and make $\angle$ CBD = BCA, &c.

See also *Creswell's* Maxima and Minima.

20. Let the straight line a be the side of the given square; straight lines b, c, denote the antecedent and consequent of the given ratio; take a fourth proportional to c, a, b, which is $a.\dfrac{b}{c}$; and find a mean proportional to a, and $a.\dfrac{b}{c}$; this mean proportional will be the side of the square required.

21. By Prop. B. Euc. vi.

$$AB.\ AC = BD.\ DC + AD^2,$$

and by Prop. 3. Euc. vi.

$$AB : AC :: BD : DC.$$

Whence

$$\frac{AC^2 \times BD}{DC} = BD.\ DC + AD^2,$$
$$= BD.\ DC + AC^2 + DC^2,$$
$$= BC \times DC + AC^2,$$
$$\therefore\ AC^2\ (BC - CD) = BC \times CD^2 + AC^2 \times CD,$$
$$\therefore\ BC\,(AC^2 - CD^2) = 2\,AC^2 \times CD,$$
$$\text{or, } 2\,AC^2 : AC^2 - CD^2 :: BC : CD.$$

TRINITY COLLEGE, 1826.

1. See *Wright's* Self-Examinations in Euc. pp. 9, and 35.

2. Euclid, i. 23. 3. Euclid, i. 41.

4. See *Wright's* Self-Examinations in Euclid, p. 46. The third case is proved by Euclid, i. 47.

5. To construct the equation (1), take the straight line

AB : linear unit, one inch for instance :: a : 1.

Produce AB to C, making BC $= 2\,b \times$ linear unit,

and from AB cut off BC$'$ $= 2\,b \times$ linear unit; then

$$AC = a + 2\,b,$$
$$\text{and } AC' = a - 2\,b,$$

which give the two linear values of c.

(2). Take AB $= a$, BC $= b$ (see Euc. vi. 13), so that AB, BC, may be in the same straight line, &c., as in Euclid.

6. Euclid, iii. 34. "Given Circle" here means that whose diameter is given. The question has no reference to position.

7. Since the angles of a quadrilateral ABCD figure are together equal to four right angles, if the sum of two opposite angles $\angle$ ADC, ABC be equal to the sum of DAB, BCD; then

$$ADC + ABC = 2 \text{ right angles},$$
$$\text{and } DAB + BCD = 2 \text{ right angles}.$$

Hence, prove that the $\angle$ BAC $= \angle$ BDC ; then describe a circle passing through B, C, D: it shall also pass through A ; for if not, let it pass through A′, and thence proceed *ex absurdo.*

8. Euclid, iv. 10.

9. On one side of the right angle, as a base, describe an isosceles $\triangle$, having the angles at that base double the angle at the vertex; bisect the $\angle$ of the $\triangle$; and, again, bisect each of these half angles; then the right $\angle$ will be quinquesected, as required.

10. Euc. v. 8. 11. Euc. v. 25. 12. Euc. vi. 14 and 15.

13. Euclid, vi. Def. 1. Similar figures are similarly situated on the sides of a right-angled $\triangle$, whose homologous sides are placed on the sides of the $\triangle$:

If A, B, C, be the areas of the similar figures, placed on the sides a, b, c, of the $\triangle$, c being the hypothenuse; then (Euc. vi. 20),

$$\begin{aligned} \mathrm{A} : \mathrm{B} &: a^2 : b^2, \\ \therefore\ \mathrm{A} : \mathrm{A}+\mathrm{B} &:: a^2 : a^2+b^2, \\ \text{and } \mathrm{C} : \mathrm{A} &:: c^2 : a^2, \\ \therefore\ \mathrm{C} : \mathrm{A}+\mathrm{B} &:: c^2 : a^2+b^2. \end{aligned}$$

But (Euclid i. 47), $c^2 = a^2+b^2$; $\therefore$ C $=$ A$+$B.

14. Sum of interior angles $= (2n-4)$ right $\angle^s$.

Sum of exterior $\angle^s = 4$ right angles.

The vertical $\angle$ of any one $\triangle = 2 \times$ interior $\angle$ of the polygon $-\pi$.

But each interior $\angle = \dfrac{2n-4}{n}$ right angles $= \dfrac{n-2}{2}\pi$,

$$\therefore \text{ each vertical } \angle = \frac{2n-4}{n}\pi - \pi.$$

$$= \frac{n-4}{n}\pi.$$

When $n = 3$; each vertical $\angle = -\dfrac{\pi}{3} = -60°$,

which is the angle vertically opposite to the interior angle of the equilateral $\triangle$.

When $n = 4$, each vertical $\angle = 0$; that is, the sides being produced, never meet; which is evidently the case with the parallel sides of a square.

In the Pentagon each vertical $\angle = \frac{\pi}{5}$, or 36°, and so on.

15. Euclid, iii. 36. Make an $\angle$ BAC, whose sides BA, AC, shall each equal half the given primeter. Draw a circle touching those sides at B and C. Take any point Q in the arc intercepted by the sides, and draw a tangent at Q, meeting the sides in P and R; then PAR is the general $\triangle$ required.

16. (1). Of the square ABCD join the diagonal BD, and make the $\angle^s$ ABP, CBQ each $\frac{1}{6}$ of a right $\angle$ or $= \frac{1}{2}$ of $\frac{1}{3}$ of right $\angle$; P and Q being in the sides AD, CD, join PQ; then BPQ is the $\triangle$ required.

(2). Bisect a side AB of the square ABCD in P; draw PQ $\perp$ AB, and trisect the right $\angle$ ABQ by PR, meeting AD in R; draw RS parallel to AB, and join PS; PRS is the $\triangle$ required.

This problem is inconsistent with Def. i. B. iv.

17. Let the chords AB, AC, drawn from the point A, cut the chord bc in b, c, which is given parallel to the tangent AT at A; join BC; then from similar $\triangle^s$ ABC, Abc,

$$AB : AC : Ac : Ab$$

$$\therefore \; AB \times Ab = AC \times Ac,$$

and similarly for any other chords.

18. Let CB be the *given* chord passing through the given point A. Join AO, O being the centre, and produce AO to the farther circumference in D, cutting also the nearer in E; then, since AC and AD are given in position, the arcs CD, BE are given. Hence the problem is reduced to that of drawing from A another chord, so that the sum of the arcs intercepted by AD, and the required chord shall = the given arc$-$(CD+BE) = arc a, suppose.

To effect this, on AO describe a segment of a circle containing an angle $= 180° -$ the $\angle \frac{a}{BO}$.

Let this segment cut the circumference in P. Join AP and produce to the circumference again in Q; AQ is the chord required. See p. 4. No. 18.

19. Let the angle ABC of the $\triangle$ ABC be double the angle ACB; draw AM $\perp$ BC, and bisect the $\angle$ ABC by BD, meeting AC in D. Draw DN $\perp$ BC.

First, prove that $NM = 2(CM-MB)$; then, from the similar $\triangle^s$ ABD, ABC, we have

$$AB : AD :: AC : AB,$$
$$\therefore AB^2 = AC \times AD,$$

and, from the similar triangles,

$$AD : NM :: AC : CM,$$
$$\therefore AB : NM :: AC^2 : AB \times CM,$$
$$:: AM^2 + CM^2 : AB \times CM,$$
$$:: AB^2 - MB^2 + CM^2 : AB \times CM,$$
$$\therefore AB^2 \times CM = AB^2 \times NM + NM \times (CM+MB)(CM-MB),$$
$$\therefore AB^2 \times BN = NM \times 2BN \times 2NM,$$
$$\therefore AB^2 = 4NM^2,$$
$$\text{and } AB = 2NM.$$

There are two cases to be considered, one in which ABC is acute, and the other in which ABC is obtuse.

20. Let the sides AB, AC, of the $\triangle$ ABC be produced to D and E; let BF, CF bisecting the exterior angles DBC, ECB meet in F (and it is easily shown they must meet). Join AF. Then prove that AF bisects BAC.

TRINITY COLLEGE, 1827.

1. (1). See Euclid, 1, Definitions and Axioms.

(2). Also, see *Wright's* Self-Examinations in Euclid, pp. 2, 3, 4, and 24; also, 5.

2. (1). Euclid, i. 32. See *Wright's* Self-Examinations in Euclid, p. 25.

(2). First, upon one side of the angle describe an equilateral $\triangle$, &c.

3. (1). Euclid, i. 35.

(2). Let P be the given point within the $\triangle$ ABC; trisect any side BC by the points D, E, and join AD, AE, PD, PE, PA. Draw AF, AG parallel to PD, PE, and join PF, PG, &c.

4. (1). Euclid, ii. 9.

(2). For, if AD bisect the base BC in D of the $\triangle$ ABC; and AM be drawn $\perp$ BC, we have

$$\begin{aligned} AB^2+AC^2 &= AM^2+BM^2+AM^2+CM^2, \\ &= 2\,AM^2+BM^2+CM^2, \\ &= 2\,AM^2+2\,BD^2+2\,DM^2, \\ &= 2\,(AM^2+DM^2)+2\,BD^2, \\ &= 2\,AD^2+2\,BD^2. \end{aligned}$$

5. (1). Euclid, ii. 13.

(2). For, if AC, BD, be the diagonals of the parallelogram ABCD, the angle at A being obtuse; then drawing DN $\perp$ BA produced, and AM $\perp$ DC, we have (Euclid, ii. 12)

$$BD^2 = AD^2+AB^2+2\,AB\times AN,$$

and $AC^2 = AD^2+DC^2-2\,CD\times DM$ (Euclid, ii. 13), &c.

6. (1). Euclid, iii. 17.

(2). For the whole problem of Tangencies, see *Wright's* Self-Examinations in Euclid, p. 137. See also p. 161, art. 337, of that work.

7. (1). Euclid, iii. 36.

(2). See p. 7. No. 15.

8. (1). Euclid, iv. 10.

(2). See *Bland's* Geometric Problems, p. 231.

9. (1). Euclid, iv. 15.

(2). Ans. 6.

10. (1). Euclid, v. 12.

(2). The sum required is a third proportional between

the first term and the difference between the first and second terms. Let the first term be AB, and the second AC, measured on the same line; then BC is this difference. From C draw CE = AB and $\perp$ AB; join BE, and draw EF $\perp$ BE, meeting BA produced in F.

Then CF represents the sum of the proportionals to infinity. See also Euclid, vi. 11.

11. (1). Euclid v. 25.

(2). See *Wright's* Self-Examinations in Euclid, p. 101.

12. (1). Euclid, vi. 3.

(2). The required Locus is a Conic Section, whose focus is C major axis along BC, vertex D, and whose directrix passes through B. It is a Parabola, Ellipse, or Hyperbola, according as CD is $=$, $>$, or $<$ DB.

13. (1). Euclid, vi. 19.

(2). Let the first part abC of the $\triangle$ ABC be made by the line ab parallel to AB; and suppose it be required by the problem that $\triangle$ abC : $\triangle$ ABC : : $m^2 : n^2$, m and n being two lines; then, take aC a fourth proportional to n, m, and AC (Euclid, vi. 12). Hence the rest of the problem is easy.

14. Euc. vi. 31.

For the Lunes of Hippocrates, see *Wright's* Self-Examinations in Euclid, p. 172.

TRINITY COLLEGE, 1828.

1. (1). The Converse of a Proposition is that proposition in which the data and quæsita of the latter are the quæsita and data of the former respectively; thus, Euclid, i. 6. is the converse of Euclid, i. 5.

The 8th, and all demonstrations *ex absurdo*, are *indirect* demonstrations; as to *contrary* propositions, the term is not to be found in any standard work.

All problems that admit of an infinite number of *different* solutions, are *indeterminate*.

Thus, if the problem be *To find two squares whose sum shall equal a given square.*

Upon the side of the given square, describe a semicircle, and taking any point in the arc of the semicircle, and joining it with the extremities of the diameter, we shall have a solution of the problem. ∴ &c.

(2). Definitions 15, 30, contain superfluous conditions, it being sufficient that a circle should have three equal radii; and that a square should be equilateral, and have one right angle. See *Wright's* Self-Examinations in Euclid on those subjects.

(3). Euclid's Definition involves the idea of distance, or the opening of a pair of compasses; whereas, this, requiring a straight line to be drawn equal to the given straight line as radius, assumes Euclid, i. 2.

2. Euclid, i. 9. See *Wright's* Self-Exam. in Euclid, p. 15.

3. Euclid, i. 11.

From any point P, without the given line, describe a circle passing through the given extremity A, and cutting the given line in another point Q; join PQ, and produce it to meet the circumference in R; join RA : it is the ⊥ required.

4. Euclid, i. 24. See Self-Examinations in Euclid, p. 21.

5. Euclid, i. 48.

Because otherwise the △ and square, although upon the same base, would not be shown to be between the same parallels; and they, therefore, could not be supposed equal.

6. Euclid, ii. 5. 7. Euclid, ii. 14. 8. Euclid, iii. 7.

For the greater segment continually decreases, and the less increases up to that position in which they become equal by Euclid, iii.

9. Euclid, iii. 20.

It is true, and proved verbatim by case 1. The only de-

duction is, that the angle BEC is greater than two right angles; or is, what is termed in Fortification, a *Salient* angle.

10. Euclid, iv. 12. 11. See *Wright's* Self-Examinations in Euclid, pp. 77 *et seq*. 12. Euclid, v. 15. 13. Euclid, v. 17.

14. From similar $\triangle^s$, AaZ, BbZ,

$$\begin{aligned} Aa : AZ &:: Bb : BZ, \\ &:: Cc : CZ, \\ &:: Dd : DZ, \\ &\&c. \end{aligned}$$

But AZ is $>$ Aa;

$\therefore$ BZ is $>$ Bb or BC; CZ is $>$ Cc, &c,

so that DZ will never be exhausted, let the series be continued ever so far.

Also, if lines from b, c, &c. be drawn parallel to AZ, it appears that

$$\begin{aligned} AZ : Aa &:: AB : Aa - Bb, \\ &:: BC : Bb - Cc, \\ &:: CD : Cc - Dd, \\ &\&c. \end{aligned}$$

$$\begin{aligned} \text{or, } AZ : AB &:: AB : AB - BC, \\ &:: BC : BC - CD, \\ &:: CD : CD - DE, \\ &\&c. \end{aligned}$$

$$\begin{aligned} \therefore AB : BC &:: BC : CD, \\ &:: CD : DE, \\ &\&c. \end{aligned}$$

$$\text{But } BC : BZ :: AB : AZ,$$

$$\therefore BC = AB. \frac{BZ}{AZ}.$$

$$\text{also, } CD = \frac{BC^2}{AB},$$

$$= AB. \frac{BZ^2}{AZ^2},$$

&c.

Observe, the sum of them is in Geometric Progression.

15. Euclid, vi. 22. 16. Euclid, vi. 25.

17. Take a straight line, a mile long, and to it apply a parallelogram that shall equal the given square (Euclid, i. 45). From either extremity of the base, with distance = a mile, describe a circle cutting the side opposite, and complete the Rhombus.

18. By Euclid, i. 47, diag.^{2} : (side)^2 : : 2 : 1,
∴ &c. See also *Leslie's* Geometry on the subject.

19. From the extremities B, C, of the base, let perpendiculars be drawn to meet the bisecting line in b and c; produce Bb, Cc, to meet the sides in B′, C′, and complete the rectangles AC, and Cb; then the sum or difference of them will equal the △, &c.

20. If B, C, be the equal angles of the base, and A the vertical angle, and P any point in the side AC, draw from P a straight line to AB, produced, so as to cut off a △ equal the given △; we must join PB, and from P draw PQ parallel to PB. Then PAQ will be the △ required.

Also PQ is greater than BC. For, since the perpendicular from A upon PQ is less than that upon BC, and the areas of the triangles ABC, APQ, are equal; ∴ base PQ is greater than the base BC.

TRINITY COLLEGE, 1829.

1. See *Wright's* Self-Exam. in Euc. p. 5. 2. Euc. i. 7. 3. Euc. i. 26. 4. See p. 13. No. 17. 5. Euc. ii. 13. 6. See *Wright's* Self-Examinations in Euc. p. 39. 7. Euc. iii. 21. 8. This depends chiefly on Euc. i. 8. 9. Euc. iii. 31. 10. Euc. iv. 5. 11. Euc. iii. 36; and see p. 5. No. 7. 12. Euc. iv. 10. 13. Euclid, v. C. 14. Euclid, v. 16.

15. The perpendicular line is evidently the shortest of the three, for with respect to the others, it subtends an acute angle, whilst they subtend a right angle. It remains, then, to

show, that the line bisecting the base is greater than that which bisects the angle.

From the angle A of the △ ABC draw AD bisecting the angle CAB, and meeting the base in D, and AE bisecting the base in E. Then, by Euc. ii. 12, and 13.

$$CA^2+AB^2 = 2\,CE^2+2\,AE^2, \text{ and (Euclid, vi. B.).}$$
$$CA.\ AB = CD.\ DB+AD^2,$$
$$\therefore\ 2\,AE^2 = AB^2+AC^2-2\,CE^2,$$
$$\text{and } 2\,AD^2 = 2\,AB.\ AC-2\,BD.\ DC,$$
$$\therefore\ 2\,AE^2-2\,AD^2 = (AB-AC)^2+2\,(BD.\ DC-CE^2),$$

See Euclid, ii. 7.

But Euclid, ii. 5.

$$BC.\ DC+DE^2 = CE^2,$$
$$\therefore\ 2\,AE^2-2\,AD^2 = (AB-AC)^2+2\,DE^2,$$
$$\therefore\ 2\,AE^2 \text{ is } > 2\,AD^2 \text{ by } (AB-AC)^2+2\,DE^2,$$
$$\therefore\ \&c.$$

16. Euclid, vi. 6. 17. Euclid, vi. 19.

18. Bisect the angle ACB of the sector by the line CD, meeting the arc AB in D, and from D draw EF ⊥ CD, and meeting CA, CB produced in E, F. Then inscribe a circle in the isosceles △ ECF; this will be the circle required.

19. Euclid, vi. 33.

TRINITY COLLEGE, 1830.

1. Euclid, i. 5. 2. Euclid i. 42.

3. They are as 18 to 20, or as 9 to 10. Euc. i. 32. cor. i.

4. Euclid, ii. 7. 5. By Euc. ii. 12, and 13.

6. Inscribe in the circle an equilateral △, and from any two of its angles draw tangents. They will meet in the point required; and its distance from the centre will thus be determined. The distance = the diameter of the circle.

7. Euclid, iii. 22. 8. Euclid, iii. 35.

9. For questions of this kind, see *Creswell's* Maxima and Minima.

10. Let the $\perp$ BE cut AC in M, and the $\perp$ CF cut AB in N, the $\perp^s$ intersecting in P. Then, since in the right-angled $\triangle^s$ PMC, PNB, the vertically opposite angles are equal; $\therefore$ the angle NBP $= \angle$ MCP. $\therefore$ AE $=$ AF.

And similarly for the other two pairs of arcs.

11. Euclid, iv. 13. 12. Euclid, v. 8. 13. Euclid, v. 12.

14. Upon AC, CB describe two segments of circles, containing each the same angle. These will intersect not only in C, but in some other point P, which will be the point required.

N. B.—The problem is indeterminate.

The locus of P is a conic section.

15. Euclid, vi. 4. 16. Euclid, vi. 18.

17. Let AM be drawn $\perp$ base BC of the $\triangle$ ABC; then Euclid, i. 47.

$$AB^2 = AM^2 + BM^2,$$
$$\text{and } AC^2 = AM^2 + CM^2,$$
$$\therefore AB^2 - AC^2 = BM^2 - CM^2; \therefore \text{Euclid, ii. 5.}$$
$$(AB+AC)\times(AB-AC) = (BM+CM)\times(BM-CM),$$
$\therefore$ &c.

18. First, construct a $\triangle$ upon an assumed base, having given the ratio of the sides, and the difference of the angles at the base. This is easily effected, by first dividing the base in the given ratio, and then making the $\triangle$ as in Question 14 of this Paper, observing that the circles must be such, that the angles of the base shall have the given difference.

This being done, describe a $\triangle$ similar to one found, but having an altitude $=$ the given perpendicular.

BOOK XI.

TRINITY COLLEGE, 1824.

1. Ans. Three points which are not in the same straight line.

2. Euclid, xi. 4. 3. Euclid, xi. 6.

4. This is a Definition, and not a Theorem. Euclid, xi. Def. 6.

5. (1). Euclid, xi. 21.

(2). There is no exception. This is also seen in Spherical Geometry.

6. (1). See *Legendre's* Geometry.

This is the fundamental proposition in Spherical Trigonometry,

$$\text{viz. } \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

(2). No; for any two of them must be greater than the third. Euclid, xi. 20.

TRINITY COLLEGE, 1825.

1. Euclid, xi. 9. 2. Euclid, xi. 11. 3. Euclid, xi. 17. 4. Euclid, xi. 21.

TRINITY COLLEGE, 1826.

1. Euclid, xi. Def. 11. 2. Euclid, xi. 6. 3. Euclid, xi. 11. 4. Euclid, xi. 17. 5. Euclid, xi. 21. 6. Euclid, xi. 28.

TRINITY COLLEGE, 1827.

1. See Euc. xi. Definitions 2. Euc. xi. 4. 3. Euc. xi. 11. 4. Euc. xi. 18.

5. Draw a plane at right angles to the straight line AB, then it will be at right angles also to the straight line CD, which is parallel to AB (Euclid, xi. 8). Then the common

section of any two planes passing through AB and CD will also be at right angles to the aforesaid plane (Euclid, xi. 19). Hence (Euclid, xi. 6), the common section is parrallel to either AB or CD.

TRINITY COLLEGE, 1828.

1. Euc. xi. 4. 2. Euc. xi. 8. 3. Euc. xi. 14. 4. Euc. xi. 17. 5. Euc. xi. 18.

6. Euclid, xi. 21. A solid angle is measured by that portion of the surface of a sphere which subtends the solid angle at its centre.

TRINITY COLLEGE, 1829.

1. Euclid, xi. def. 6. The inclination of two straight lines not lying in the same plane is that acute angle which is formed by drawing from any point in one of them a straight line parallel to the other.

2. Euclid, xi. 5. 3. Euclid, xi. 10. 4. Euclid, xi. 20.

5. The projection of a straight line on a plane is the intersection of a plane passing through the straight line $\perp$ to the plane of projection.

Hence, the plane which projects the straight line, which is perpendicular to one plane, upon a second plane, is perpendicular to both planes, and, therefore, to their intersection.

6. Let AB, CD, be the given lines. Join AC, and complete the parallelograms ACDE, ACFB; then complete the parallelogram FCDG; then complete the parallelogram EDGH, and join BH. The figure will be the parallelopided required.

ST. JOHN'S COLLEGE, 1825.

1. Let C be the centre of the circle, and DF touch the circle in E. Join CD, CE, CF. Then prove the $\angle$ ECF $= \angle$ FCB, and $\angle$ ECD $= \angle$ DCA, &c.

2. Euclid, v. 12.

3. Let ABC be an isosceles $\triangle$, having $AB = AC$. Draw $BM \perp AC$; then, Euclid, ii. 13.

$$\begin{aligned} BC^2 &= 2\,AC^2 - 2\,AC \times AM, \\ &= 2\,AC \times CM, \text{ (Euclid, ii.)} \\ \text{or } BM^2 + CM^2 &= 2\,AM \times CM + 2\,CM^2, \text{ (Euclid, ii.)} \\ \therefore\ BM^2 &= 2\,AM \times CM. \end{aligned}$$

4. For if the polygon consist of $2n$ sides, it may be divided into n quadrilateral figures, all *inscribed* in the circle. Whence (Euclid, iii. 22) the proof is easy.

5. Let AB, AC, be the equal sides of the $\triangle$. From A with radius $AP > \frac{1}{3}AB$ but $< \frac{1}{3}(AB+BC)$ describe a circle, and from B with radius $= 2\,AP$ describe another intersecting the former in P. Then P is the point required.

6. Join the diagonals, and make use of Euclid, ii. 12 and 13.

7. Bisect each of the sides of the $\triangle$ in P, Q, R; and join PQ, PR, QR. Then PQR is the $\triangle$ required.

8. First prove that the $\triangle^s$ IED, IHF, ICE, IFG, are similar.

9. From similar $\triangle^s$,

$$\begin{aligned} Bp &: AB :: BC : AC, \\ pq &: Bp :: BC : AC, \\ qr &: pq :: BC : AC, \\ &\text{\&c.} \end{aligned}$$

$$\therefore\ Bp + pq + qr + \text{\&c.} = \frac{AB.BC}{AC}\left(1 + \frac{BC}{AC} + \frac{BC^2}{AC^2} + \ldots \text{to } \infty\right)$$

$$= \frac{AB.BC}{AC - BC},$$

$$\therefore\ Bp + pq + \ldots\ldots : BC :: AB : AC - BC.$$

10. Euclid, xi. 20.

ST. JOHN'S COLLEGE, 1828.

1. Euclid, v. 13. 2. Euclid, vi. 15.

3. This is easy, after knowing Euclid, vi. 15.

4. Euclid, xi. 20.

5. First, describe any $\triangle$ having each of the angles at the base double of the vertical angle, Euclid, iv. 10.

Then, at the centre of the circle, make two angles, which shall equal the supplements of the equal angles of this △, by straight lines drawn to the circumference in P, Q, R. At P, Q, R, draw tangents. These will form the △ required.

6. Since the base and area of the △ are given, ∴ the altitude is given. Consequently, place this base so that its extremities may lie on adjacent sides of the square. From one extremity draw a perpendicular to the line, and make it = the altitude of the △. From the upper extremity of this ⊥ draw a line parallel to the base, and meeting a third side of the square, &c. &c.

7. That is the greatest whose base is the side of the square, and whose vertex is in the opposite side of the square, and this = half the square.

8. Let the circles cut in A, B. Join their centres C, D, the line CD cutting the circumferences in E, F. From E draw EG meeting the circle EA in G. Join GF. Inflect FH in the circle AF, making it = GE. Join EH. Then GH is the parallelogram required. Find the limitations.

9. From A, the centre of the circle, draw AC = given straight line, and making an angle with the *given* radius BA produced, equal to the given angle. From C draw CD parallel to AB, and meeting the circumference in D. Then D is the point required. Find the limitations.

10. If the given lines AB, CD, are parallel, there is no difficulty. If not, produce them to meet in E. In EA, EB, take respectively EF, EG in the given ratio. Draw FH, GK ⊥ AE and EB to meet in L. Join EL. Then EL is the line required.

11. Divide the base AB in E, so that AE : EB may be the given ratio. Then determine a point D in the ⊥ CD, such that ED may bisect the angle ADB. See p. 15. No. 14.

12. Let the chord AB be produced both ways from A to C, and from B to D, making AC = BD. From C, draw

CE touching the circle in E, and bisect AB in F. Join EF, and produce it to the circumference in G. Join DG. Then prove that DG = CE, and is therefore a tangent.

To effect this, take O the centre of the circle. Join OC, OF, OD, OE, OG, and use the $\triangle^s$ COE, DOG.

13. Let BAC be the given vertical angle, then, drawing CM $\perp$ AB, the ratio AM : CM is known $= r$, suppose.

But if m^2 be the sum of the squares of the sides containing the vertical angle, and c the unknown side opposite the $\angle$ C, and a the base; then, Euc. ii. 13,

$$a^2 = m^2 - 2c \times \text{AM}.$$

Also, if α^2 be the given area of the $\triangle$,

$$2\alpha^2 = c \times \text{CM},$$

$$\therefore\ a^2 = m^2 - 4r \times \alpha^2.$$

Take $\therefore$ a semicircle on the diameter PQ $= m$, and inflect a straight line PR $= 2\sqrt{r}.\ \alpha$. Join QR. QR is the base of the $\triangle$. On QR describe a segment of a circle containing an $\angle$ equal = the given angle $\angle$, and from P draw PS $\perp$ PQ, making it $= \dfrac{2\alpha^2}{a}$, and draw ST meeting the circumference in T. Join PT and QT. The $\triangle$ PQT is that required.

All this may be easily put into Geometrical form.

14. See *Creswell's* Maxima and Minima.

CORPUS CHRISTI COLLEGE, 1825

1. Euc. i. 47. 2. Euc. ii. 10. 3. Euc. ii. 13. 4. See p. 5. No. 7.

5. Let AB be the straight line. Bisect it in C. Draw BD $\perp$ AB, and make it equal to the side of the given square. Join CD, and in AB produced take CP = CD. Then,

$$\text{since PB} \times \text{PA} + \text{BC}^2 = \text{PC}^2. \quad (\text{Euc. ii. 6.})$$

$$\text{and PC}^2 = \text{CD}^2 = \text{BC}^2 + \text{BC}^2,$$

$$\therefore\ \text{PB} \times \text{PA} = \text{BD}^2.$$

See, also, *Bland's* Geom. Problems, p. 156.

6. Euc. ii. 10.

7. Let AB be the *position* of the base; CD parallel to AB, and distant from it by the given radius; then CD is the locus

of the centre of the inscribed circle. From centre A, at distance = given difference, describe a circle, &c. The locus is a circle.

8. Euc. vi. 4. 9. Euc. vi. 19. 10. Euc. vi. B.

CORPUS CHRISTI COLLEGE, 1826.

1. Euc. i. 21. 2. Euc. i. 44. 3. Euc. i. 48. 4. Euc. ii. 9. 5. Euc. iii. 13. 6. Euc. iii. 35. 7. Euc. iv. 12. 8. Euc. vi. 19. 9. Euc. vi. D.

10. Take CB : AC :: AC : CP, and P will be the point required.

11. Let CAB be the △, having the obtuse angle CAB. About it describe a circle whose centre is O. Join OA, and upon it describe a semicircle intersecting the base BC in D. Join AD. Then AD is a mean proportional between CD, DB. For, join OD, and produce AD to the circumference in E, &c. &c. It is a necessary limitation, that the angle shall be either right or obtuse. Why?

12. This is Euc. ii. 11 in disguise.

CORPUS CHRISTI COLLEGE, 1827.

1. Euc. i. 5. 2. Euc. i. 35. 3. Euc. i. 27. 4. Euc. ii. 13. 5. Euc. iii. 9. 6. Euc. iii. 21. 7. Euc. iii. 35. 8. Euc. iv. 10. 9. Euc. v. 25. 10. Euc. vi. 3. 11. Euc. vi. 19. 12. Euc. vi. 31. 13. Euc. ii. 12 and 13.

14. This problem is indeterminate.

From an assumed point D, in the side AB of the △ ABC, draw DE parallel to the base BC. Make DF : DE in the given ratio, and from D describe an arc cutting BC in F, and complete the parallelogram whose sides are ED, DF.

15. Let AB be the given base, DC the position of the bisecting line produced to meet the base in C. Draw BE ⊥ CD, and produce it making EF = EB. Join AF, and produce it to meet CD produced in G, and join GB. Then AGB is the △ required.

16. Let ABCD be the trapezium; bisect the opposite sides AB, CD in E, F. Join EF, and bisect it in P. Join PA, PB, PC, PD. Take any other point P′, and join P′A, P′B, P′C, P′D, P′E, P′F.

Then prove that

$$PA^2+PB^2+PC^2+PD^2 = 2\,AE^2+2\,CF^2+4\,PF^2,$$
$$P'A^2+P'B^2+P'C^2+P'D^2 = 2\,AE^2+2\,CF^2+4\,PF^2+4PP'^2,$$

then, the excess of the latter four squares above the former is $4\,PP'^2$, which more than proves the prop.

CORPUS CHRISTI COLLEGE, 1830.

1. See p. 11. 2. See *Wright's* Self-Examinations in Euc. p. 4. 3. See *Wright's* Self-Examinations in Euc. p. 5. 4. Euc. i. 5. 5. Euc. i. 13. 6. Euc. i. 19. 7. Euc. i. 32. See *Wright's* Self-Examinations in Euc. p. 25. 8. Euc. i. 40. 9. Euc. ii 1. See *Wright's* Self-Examinations in Euc. p. 39. 10 Euc. ii. 13. 11. Euc. iii. 20. See *Wright's* Self-Examinations in Euc. p. 55. 12. Euc. iii. 22. 13. Euc. iii. 35. 14. Euc. iv. 13. 15. See *Wright's* Self-Examinations in Euc. p. 75. 16. Euc. v. i. 17. Euc. v. 8. 18. Euc. vi. 4. 19. Euc. vi. 19. 20. Bisect its three sides, &c.

21. First, if the sides opposite A, B, C, be called a, b, c, A being the right angle; then, prove that

$$\frac{R\,(a+b+c)}{2} = \frac{AD \times a}{2},$$

thence that

$$R = \frac{bc}{a+b+c},\ r = \frac{AD \times DB}{c+AD+DB},\ r' = \frac{AD \times DC}{b+AD+DC},$$

$$\text{thence get } r = \frac{bc^2}{a\,(a+b+c)},\ r' = \frac{cb^2}{a\,(a+b+c)}.$$

$$\text{Hence } r^2+r'^2 = \frac{b^2c^2\,(c^2+b^2)}{a^2\,(a+b+c)^2} = \frac{b^2c^2}{(a+b+c)^2} = R^2,\ \therefore \&c.$$

Observe. The circle inscribed in the △ ABC may be hence proved to equal the sum of the other two.

22. Upon the side describe two segments of circles, containing angles = the given ∠ and = double the given ∠. Inflect a straight line = the sum of the sides, &c

CHRIST'S COLLEGE, 1828.

1. Euc. i. 16. 2. Euc. i. 45. 3. Euc. i. 48. 4. Euc. ii. 6. 5. Euc. iii. 17. 6. Euc. iii. 32. 7. Euc. iii. 37. 8. Euc. iv. 4. 9. Euc. iv. 11. 10. Euc. vi. 8. 11. Euc. vi. 15.

MISCELLANEOUS QUESTIONS.

12. The product is .000392.

13. Ans. $\frac{7}{16}$ is $> \frac{2}{5}$ by $\frac{3}{80}$.

14. Ans. 9*s.* 11$\frac{1}{4}$*d.*

15. Ans. 14*l.* 5*s.*

16. His expenditure exceeds his income by 38*l.* 5*s.* a year. He is in debt 229*l.* 10*s.*

17. $\sqrt{31595641} = 5621$,
$\sqrt[3]{175616} = 56$.

18. The quotient is 16*l.* 4*s.* 9*d.* $\frac{1}{7}$.

19. The quotient is $\dfrac{1}{2a} + \dfrac{x}{4a^2} + \dfrac{x^2}{8a^3} + \text{\&c.}$
The product is $mnx^3 - (np+mp+m)x^2 + (nq+p^2+p)x - pq - q$.

20. The root is $ax - b + \frac{3}{2}$.

21. The value is $\frac{71}{99}$.

22. The G. C. M. is $a - 3b$.

23. The first sum is 272; the second is $\dfrac{4^7-1}{3}$, or 5461.

24. See *Wood*, art. 253.

25. (1). $\dfrac{x}{5-x} = 4$, and $-\frac{1}{4}$, whence x.

(2). $x = 9$.

(3). $x = 0$, and $\sqrt{3}$, and $-\sqrt{3}$.

(4). Clearing of fractions, &c.

$$x^2 - 12x = -32,$$
$$\therefore x = 8 \text{ and } 4.$$

(5). From the first $\sqrt{(x+y)} = 3$, and -6,

$$\therefore x+y = 9 \text{ and } 36.$$

Whence, by means of the second, the question is easy.

(6). $x = 7$, $y = 10$, $z = 9$.

ST. PETER'S COLLEGE, 1825.

1. Euc. i. 4. 2. Euc. i. 9. 3. Euc. ii. 14. 4. Euc. iii. 3. 5. Euc. iv. 6. 6. Euc. v. 15. 7. Euc. vi. 25. 8. Euc. xi. 5.

9. To the given base apply a parallelogram equal to double the given △ (Euc. i. 44); then bisect the given base, and from its middle point draw a ⊥ meeting the side or sides produced of the parallelogram in a point. This point will be the vertex of the required △.

10. With the given radius describe a circle; then drawing any radius OP, at P, draw a tangent BPA; at O the centre, make the angle POA = a right ∠ − ½ given ∠ A. Also, in like manner, on the other side of PO, make the angle POB = right ∠ −½ B, and from A and B draw tangents meeting in C; then ABC is the △ required.

11. From the point A, with a radius equal a side of the given square, describe an arc cutting the given circle in C; then join BC, and produce it to meet the tangent in P. P is the point required.

When the side of the square is greater than the denominator of the ⊙, the problem is impossible.

12. Upon the straight line describe a ½ ⊙, and from either extremity draw a straight line, making an ∠ = ½ a right ∠, and meeting the circumference in P. From P draw PM ⊥ to the diameter. M will be the required point of section. There are two such points.

13. If r be the radius of the ⊙, s the side of the square, and o that of the octagon; then, taking the similar right-angled △ˢ in a ½ ⊙, whose sides are o, $r-\frac{s}{2}$; and $2r$, 0, we get $2r : 0 :: 0 : r - \frac{s}{2}$.

Again, C being the centre of the ⊙, CA its radius, AB the side of the hexagon = CA = r; then produce AB to P, so

that AP $=$ side of the square $= s$, and draw PQ touching the $\odot$ in Q; then, since

$$PQ^2 = PA \times PB = (AP - AB) \times PA = (s-r) \times s = s^2 - rs,$$

$$\therefore\ PQ^2 = 0^2, \text{ and } PQ = 0.$$

14. From the given point P draw a tangent to the $\odot$ in Q, and let R : 1 be the given ratio; then take x a point in the concave circumference, such that

$$Px^2 : PQ^2 :: R+1 : R, \text{ and } x \text{ is the point required.}$$

15. Divide the base BC$=a$, in the ratio of $m : n$; upon a describe a $\frac{1}{2}\odot$, and at D erect the $\perp$ DE, meeting the circumference in E. Then, with B as centre and distance $=$ BE, describe an arc cutting the base a in A′. From A′ draw A′B′ parallel to C.

The proof depends upon similar $\triangle^s$ being in the duplicate ratio of their homologous sides.

16. From the points where the parallels meet the plane, draw $\perp^s$ to the plane. The $\perp^s$ are parallel (Euc. xi. 6). Hence we have two lines which meet parallel to two others which meet. They, therefore, contain equal angles (Euc. xi. 10). Hence, &c.

17. See *Woodhouse's* Trig., or *Hind's*.

18. See *Woodhouse's* Trig., or *Hind's*.

Also, if A $= 18°$; then $2 \sin 18 . \cos 18 = \sin 36 = \cos 54$,

$$\text{whence } \sin 18° = \frac{\sqrt{5}-1}{4}.$$

19. Since the ratio of the sines of the angles A, B, C, is the same as that of the opposite sides a, b, c; let

$$a : b : c :: m : n : r,$$

and suppose $a+b+c = p$; then $b = a\,\dfrac{n}{m}, c = a.\dfrac{r}{m}$,

$$\therefore a\left(1 + \frac{n}{m} + \frac{r}{m}\right) = p, \text{ whence } a, \text{ and } \therefore b, \text{ and } \therefore c.$$

20. If A be the vertical angle, B, C the others, a, b, c,

the opposite sides, and s, s' the segments of a, contiguous to B, C; then (Euc. i. 47).

$$c^2-s^2 = (\text{perp.})^2 = b^2-s'^2,$$
$$\therefore\ s^2-s'^2 = c^2-b^2,$$
$$\text{and } s+s' : c+b :: c-b : s-s'.$$

21. From centre C, and radius CA $= 1$, describe a $\frac{1}{2}\odot$ ABD making AB $= \angle$ A. Bisect $\angle$ BCA by CT meeting the tangent at A in T, join DB, and draw BM $\perp$ AC. Then, from similar $\triangle^s$ ACT, MDB.

$$\text{AT}^2 : \text{AC}^2 :: \text{BM}^2 : \text{DM}^2 :: \text{AM}\times\text{DM} : \text{DM}^2,$$
$$:: \text{AM} : \text{DM},$$
$$\text{or } \tan^2\frac{\text{A}}{2} : 1 :: 1-\cos\text{A} : 1+\cos\text{A}.$$

Also, produce AT to T$'$, making TT$'$= CT, and join CT$'$; then

$$\tan\text{A}+\sec\text{A}=\text{AT}'=\tan\text{ACT}' = \tan\left\{\text{A}+\tfrac{1}{2}\left(\frac{\pi}{2}-\text{A}\right)\right\},$$
$$= \tan\left(\frac{\pi}{4}+\frac{\text{A}}{2}\right) = \&\text{c}.$$

22. At the first point of observation, let the elevation be θ, the distance from the first to the second d, at which let the elevation be θ'; from the second to the third let the distance be d'; and the third elevation suppose θ''; then the required height will be found to be

$$\sqrt{\frac{dd'(d+d')}{d'\cot^2\theta+d\cot^2\theta''-(d+d')\cot\theta'}}.$$

23. Let $x-1$, x, $x+1$ be the sides of the $\triangle$, θ, $\pi-3\theta$, 2θ, its angles opposite to these,

$$\text{then } \frac{x-1}{x}=\frac{\sin\theta}{\sin 3\theta}=\frac{\sin\theta}{\sin\theta\cos 2\theta+\cos\theta\sin 2\theta}=\frac{1}{\cos 2\theta+2\cos^2\theta}$$
$$\text{and } \frac{x-1}{x+1}=\frac{\sin\theta}{\sin 2\theta}=\frac{1}{2\cos\theta}.$$

Hence $x=5$, and $\theta=\cos^{-1}\frac{3}{4}$, which may be found by the tables.

24. Sin 2^n A $= 2\sin 2^{n-1}$ A. $\cos 2^{n-1}$ A $= 2^2.\sin 2^{n-2}$ A $\times$ $\cos 2^{n-2}$ A $\cos 2^{n-1}$ A, &c.

For the series, which are both *recurring*, see *Woodhouse* or *Hind*.

ST. PETER'S COLLEGE, 1828.

1. Euc. i. 20. 2. Euc. v. 12. 3. Euc. iv. 11. 4. Euc. xi. 11. 5. Euc. iii. 27. 6. Euc. vi. 8. 7. Euc. ii. 11. 8. Euc. v. 25 9. Euc. ii. 14. 10. Euc. vi. 25. 11. Euc. i. 48. 12. Euc. xi. 6

ST. PETER'S COLLEGE, 1830.

1. Euc. v. 16. 2. Euc. v. 23. 3. Euc. vi. 5. 4. Euc. vi. 25.

5. *Woodhouse's* Trigonometry.

6. Let $\theta+\phi$ be the French angles; then

$$\begin{aligned} \theta+\phi &= \frac{10}{9}\times 75 = \frac{250^\circ}{3} = 83^\circ.\ 333333\ \&c. \\ \theta-\phi &= 26^\circ.\ 333333\ \&c. \\ \therefore\ \theta &= 54^\circ.\ 83'33''\ldots\ldots \\ \phi &= \frac{57^\circ}{2} = 28^\circ.\ 50'. \end{aligned}$$

Hence, English $= \frac{9}{10}\times$ French are found.

7. See *Woodhouse's* Trigonometry.

8. $$\begin{aligned} \text{Cosec}\,3\,\theta &= \frac{1}{\sin 3\,\theta} = \frac{1}{\sin\theta\ (4\cos^2\theta - 1)}, \\ &= \text{cosec}\,\theta.\ (1+2c)^{-1}, \\ &= \text{cosec}\,\theta\ (1-2c+2^2c^2+\&c.) \end{aligned}$$

9. $$\begin{aligned} 2\sec\frac{\pi-A}{3}\ \sin\frac{\pi+A}{3} &= \cos\frac{2\,A}{3} - \cos\frac{2\pi}{3}, \\ &= \cos\frac{2\,A}{3} + \frac{1}{2}, \end{aligned}$$

$$\therefore\ 4\sin\frac{A}{3}.\ \sin\frac{\pi-A}{3}.\ \sin\frac{\pi+A}{3} = 2\sin\frac{A}{3}.\ \cos\frac{2\,A}{3} + \sin\frac{A}{3}.$$

$$= \sin A - \sin\frac{A}{3} + \sin\frac{A}{3} = \sin A.$$

10. If a be one of the equal sides, the hypothenuse $= a\sqrt{2}$,

$$D = \frac{a\sqrt{2}}{1+\sqrt{2}},\ P = a\sqrt{2}.\ (1+\sqrt{2}),\ \therefore\ \&c.$$

11. If A, B be the objects, and C the given position; a, b the linear distances, and C the difference of the given angular distances; then, $x^2 = (a-b)^2.\ \sec^2\theta$,

$$\text{in which } \tan^2\theta = \frac{2ab}{(a-b)^2}\ \text{vers. C}.$$

12. See *Woodhouse's* Trigonometry.

13. $\text{Sin A. sec}\,\dfrac{A}{2} = 2\sin\dfrac{A}{2}$,

$$\sin\frac{A}{2}\,.\,\sec\frac{A}{2^2} = 2\sin\frac{A}{2^2}, \text{ \&c.}$$

$$\therefore \sin A.\ \sec\frac{A}{2}.\ \sec\frac{A}{2^2}\ldots\sec\frac{A}{2^n} = 2^n.\ \sin\frac{A}{2^n}.$$

But, ultimately, when $n = \infty$, $\dfrac{A}{2^n} = \sin\dfrac{A}{2^n}$, $\therefore$ &c.

Also, since $\tan\theta = \cot\theta - 2\cot 2\theta$,

$$\therefore \frac{1}{2}\tan\frac{A}{2} = \frac{1}{2}\cot\frac{A}{2} - \cot A,$$

$$\frac{1}{4}\tan\frac{A}{4} = \frac{1}{4}\cot\frac{A}{4} - \frac{1}{2}\cot\frac{A}{2}, \text{ \&c.}$$

$$\frac{1}{2^n}\tan\frac{A}{2^n} = \frac{1}{2^n}\cot\frac{A}{2^n} - \frac{1}{2^{n-1}}\cot\frac{A}{2^{n-1}},$$

$$\therefore \text{the sum} = \frac{1}{2^n}\cot\frac{A}{2^n} - \cot A,$$

and the sum to ∞ is $\dfrac{1}{A} - \cot A$ (by Vanishing Fractions).

QUEEN'S COLLEGE, 1830.

1. Join the two given points, A, B, and bisect AB in C. Draw CD $\perp$ AB, and meeting the given line in D. D is the point required.

2. Prove the proposition *ex absurdo*, saying, if the alternate angles AGH, GHD (Euclid's fig.) be not equal, make the $\angle$ GHK = $\angle$ AGH; then, (by *Playfair's* axiom) HK, when produced, will make a $\triangle$ with GH and AB, two of whose angles are equal to two right angles, &c. &c.

3. See *Bland's* Geometrical Problems, p. 131.

4. See *Bland's* Geometrical Problems, p. 126.

5. See *Bland's* Geometrical Problems, p. 143.

6. Let the sides BC, CA, AB, of the $\triangle$ ABC, be bisected in a, b, c, respectively. Join Aa, Bb, Cc. These intersect in P (a well known deduction, for which see *Bland*).

And it may be easily shown,

$$\therefore 4\,(AB^2+AC^2+BC^2) = 4\,(Aa^2+Bb^2+Cc^2)+AB^2+AC^2+BC^2.$$

But $Aa = \frac{3}{2}.\ AP$, $Bb = \frac{3}{2}.\ BP$, $Cc = \frac{3}{2}.\ CP$,

$$\therefore AB^2+AC^2+BC^2 = 3\,(AP^2+BP^2+CP^2).$$

7. Upon the side AB of the given square, describe a semi-⊙, and make a square, whose side is CD, equal the given rectangle; from A, draw AE ⊥ AB, and = CD, and from E, draw EF parallel to AB, meeting the circle in F. Join AF, BF, and AF, BF, are the straight lines required.

8. If AB be the hypothenuse of the right-angled △ ABC, take D in AB, so that DB : BC :: AB : 2 AB+BC. For, &c.

9. In the side CB of the △ ABC, take CD : CB :: 1 : $\sqrt{2}$, :: side of a square : to its diagonal. Draw DE parallel AB. Then the △ is bisected by DE.

10. The area of the △, and its base, are given in magnitude. Hence, its altitude is given. Upon the given straight line AB, describe, therefore, a segment of a circle containing an ∠ = the given angle; draw AC ⊥ AB and = the altitude, and draw CD parallel to AB, meeting the circumference in D. Join AD, BD; ABD shall be similar and equal to the △ required. Whence the △ is easily determined also in position.

11. Bisect the quadrant, draw tangent at point of section, making a △ with the radii produced, and inscribe in the △ a circle. It is the one required.

12. AB, CD, being the straight lines, take any point E in the line CD, and draw EF parallel to AB. Pass a plane through CD ⊥ plane DEF, and meeting the line AB in G. Draw GH ⊥ CD, and GH is the line required. It is also the shortest distance between the two lines AB and CD.

13. The edges are all equal; and the altitude $= \sqrt{\frac{2}{3}} \times$ edge.

14. The direction of the cutting plane being given along the side, it will cut the edge opposite the angle through which

it passes, in a given point. Hence, the question is reduced to that of *bisecting a* $\triangle$ *by a line drawn from a given point in one of its sides.*

15. The distance required is (see *Wright's* Self-Examinations in Euclid, p. 128),

$$PP' = \sqrt{\{(a-a')^2+(b-b')^2+(c-c')^2\}}.$$

EMMANUEL COLLEGE, 1821.

1. Let ABCD be the trapezium, AB being parallel to CD. Bisect BC in E, join AE, ED. Then, AED $= \frac{1}{2}$ the trapezium. For, draw BF $\perp$ DC, and EG $\perp$ DC.

Then, trapezium $= AB \times BF + DC \times BF$,

$\triangle ABE + \triangle DEC = AB \times EG + DC \times EG$, &c. &c.

2. By Euc. ii. 12, and 13.

3. Let A be the given point, B that in the circle. Find the centre C, draw BD touching the $\odot$ in B. Join AB, and on AB describe a segment of a circle, containing an $\angle$ equal the $\angle$ DBA. This segment touches the given circle in B, and passes through A.

4. For the angle in a quadrant $=$ a right $\angle + \frac{1}{2}$ a right $\angle$, &c. &c.

5. See p. 23. No. 22.

6. See p. 21. No. 11.

7. For a diameter of the circle is the diagonal of the inscribed square.

8. This depends upon Euclid, iii. 36.

9. The centre of the circle is the intersection of the diagonals, and the radius is the perpendicular drawn from the centre upon any one of the sides.

10. Making any isosceles $\triangle$ ABC, with the given vertical $\angle$ A, and in the side AB produced, take $\triangle$ ABC : scalene $\triangle$: : $AB^2 : AB'^2$, and complete the isosceles $\triangle$ AB′C′, &c.

11. Let ABC be the isosceles $\triangle$ having AB = AC. Take

any point D in BC, and produce AD to meet the circumference in E; and draw AM $\perp$ BC.

$$\begin{aligned} \text{Then, } AE \times AD &= AD^2 + AD \times DE \text{ (Euclid, ii.)}, \\ &= AM^2 + DM^2 + BD \times DC, \\ &= AM^2 + BM^2 \text{ (Euclid, ii.)}, \\ &= AB^2. \end{aligned}$$

12. See *Wright's* Self-Exam. in Euclid, p. 159., art. 329.

EMMANUEL COLLEGE, 1825.

1. Euc. i. 4. 2. Euc. i. 44. 3. Euc. ii. 6. 4. Euc. ii. 13. 5. Euc. iii. 15. 6. Euc. iii. 25. 7. Euc. iii. 32. 8. Euc. iv. 8. 9. Euc. iv. 11. 10. Euc. v. 25. 11. Euc. vi. 4. 12. Euc. vi. 19.

---

1. The figure is equilateral. The angles which two of its sides make with a side of the square are together equal a right angle, and $\therefore$ the figure is rectangular.

2. In the side AB, let the given point be P; then, supposing ab to be the side of the parallelogram opposite to AB, a being opposite to A, in ab, take bp = AP, and join Pp; Pp will divide the parallelogram as required.

3. Let ABC be the $\triangle$, C being the right $\angle$. Also, CD be the bisecting line, and CE the $\perp$ meeting the base in D and E. Then $\angle$ ECD + $\angle$ EDC = right angle. But (if AC be less than CB), $\angle$ EDC $= \frac{1}{2}$ right $\angle$ + B. Whence $\angle$ ECD $= \frac{1}{2}$ (A$-$B). Which shows the proposition wrong.

4. Let A be the vertex of the isosceles $\triangle$, B and C the angles at the base. Then, taking D any point in the base, and AM a $\perp$ upon it, by Euclid, xii. 2, we have $AB^2 = AD^2 + BD^2 + 2\,BD \times DM = AD^2 + BD \times DC$.

5. This is only a particular case of the Apollonian Tangency, "To describe a $\odot$ passing through a given point and touching a given straight line and a given circle." (See Self-Examinations in Euclid, p. 163.)

6. For the straight line joining the point of contact and the

common centre is $\perp$ to the tangent; *i. e.* $\perp$ to the straight line placed in the outer circle. It is $\therefore$ bisected by Euc. iii. 3.

7. Take a straight line AB = the given sum, and from B draw BC $\perp$ AB and = the given mean proportional. Then, upon AB describe a $\frac{1}{2}\odot$, and from C draw CD parallel to AB, and meeting the $\odot$ in D; from D draw DE $\perp$ AB, meeting it in E; then, AE, EB, are the lines required.

8. Ans. Six.

9. Let AB be the given difference; on AB, as diagonal, describe a square ACBD; in AB take AE = difference of AB and AC; also, in AB produced, take AF : AB :: AB : AE, and upon AF, as diagonal, describe a square. This shall be the square required.

10. For the diagonal of the inscribed square = side of the circumscribed square.

EMMANUEL COLLEGE, 1827.

1. Euc. i. 24. 2. Euc. i. 44. 3. Euc. ii. 7. 4. Euc. iii. 13. 5. Euc. iii. 33. 6. Euc. iv. 4. 7. Euc. iv. 11. 8. Euc. v. 25. 9. Euc. vi. 5. 10. Euc. vi. 31.

1. Let ABC be the $\triangle$; bisect AB, BC, in D, E, and draw the perpendiculars DP, EF, meeting in F. Draw FG $\perp$ AC, and prove that AG = GC. See Euclid, iv. 5.

2. By Euc. ii. 13, and 14

3. See *Wright's* Self-Exam. in Euclid, p. 161. No. 337.

4. See *Wright's* Self-Exam. in Euclid, p. 163. No. 341.

5. Let ABC be the $\triangle$, and P a given point in the side AB; bisect the base in D; join AD, PD. Draw AE parallel to PD, meeting BC in E. Join PE. The $\triangle$ ABC is bisected by PE.

ARITHMETIC AND ALGEBRA.

TRINITY COLLEGE, 1825.

[P. 37.]

1. Since $2l.\ 10s. = 2\frac{1}{2}l. = \frac{5}{2}l.$ and $3s.\ 6d. = \frac{7}{40}l.$

$$\therefore \frac{7}{40} \div \frac{5}{2} = \frac{2 \times 7}{5 \times 40} = \frac{7}{100} = .07.$$

2. *Ans.* The factors are of the form 2^n, 5^m; for these being multiplied by 5^n and 2^m will become 10^n and 10^m respectively, which are the denominators of decimals.

3. The work is

$$\begin{array}{r} 54.\ 6 \\ 6.\ 93 \\ \hline 1\,4\,1\,6 \\ 4\,0\,4\,6 \\ 2\,8\,3\,0 \\ \hline 3\,0\,4.\ 8\,7\,6 \\ \hline \end{array} \qquad \begin{array}{ccc} & 6 & \\ 4 & \times & 7 \\ & 6 & \end{array} \text{ proof by casting out elevens.}$$

304.876 square-feet expressed in the decimal scale.

By duodecimals :

	Feet.	Inches.		
	54.	6		
	6.	9.	3^{12ths}	
	327.	0		
	40.	10.	6	
	1.	1.	7.	6
sq. feet	369.	0.	1.	6,

or 41 sq. yards. 0 feet. $1^{12th.}$ $6^{144ths.}$

See *Bonnycastle's* Arith.

4. Number of cannon $\propto \dfrac{\text{men killed}}{\text{time} \times \text{number of rounds in a minute}}$.

$$\therefore 10 : \frac{270}{1\frac{1}{2} \times \frac{3}{5}} :: x : \frac{500}{1 \times \frac{5}{6}},$$

$$\text{or, } 1 : \frac{27 \times 10}{9} :: x : 600,$$

$$\therefore x = \frac{60}{3} = 20.$$

5. The gain, at 10 per cent., upon $25l.$ is $50s.$ or $2l.$ $10s.$

$\therefore$ 30 gallons must be sold for $27l.$ $10s.$ or

$$\therefore \frac{27l.\ 10s.}{30} = \frac{550}{30}s. = 18s.\ 4d. \text{ the answer.}$$

6. $(a^2+ax+x^2)(a^2-ax+x^2)=(a^2+x^2+ax)(a^2+x^2-ax)$ $=(a^2+x^2)^2-a^2x^2$ (*P. T.* vol. i. p. 70) $=a^4+2a^2x^2+x^4-a^2x^2$ $=a^4+a^2x^2+x^4$.

7. The quotient of $a-x$) a (is $1+\dfrac{x}{a}+\dfrac{x^2}{a^2}+\ldots$ to ∞.

Since the quotient is a geometric series whose common ratio is $\dfrac{x}{a}$, it is arithmetically equal to $\dfrac{a}{a-x}$ when a is $> x$; for such convergent series $= \dfrac{1}{1-\dfrac{x}{a}} = \dfrac{a}{a-x}$.

8. *Wood*, art. 90, and seq.

9. The principle of indices in b^n denotes a continued product of the same letter b, and is assumed; hence it is assumed that b^2 denotes $b \times b$. If $b^2 = a$; since the operation of extracting the square root of b^2 is the inverse of $b \times b$ or b^2, we analogously assume the root of b^2 to be $(b^2)^{\frac{1}{2}} = a^{\frac{1}{2}}$.

10. The denominator of (α) is erroneously printed. The fraction should be

$$\frac{x^2+(a-b)\,x-ab}{x^2+(a+b)\,x+ab} = \frac{(x+a)\,(x-b)}{(x+a)\,(x+b)} = \frac{x-b}{x+b}.$$

(β) gives

$$\frac{x}{1-x}\left\{1-\frac{x}{1-x}+\frac{x^2}{(1-x)^2}\right\} = \frac{x}{1-x}\left\{1+\frac{2x^2-x}{(1-x)^2}\right\}$$

$$= \frac{x}{(1-x)^3}.\,(1-3\,x+3x^2) = \frac{x\{(1-x)^3+x^3\}}{(1-x)^3} = x+\frac{x^4}{(1-x)^3}.$$

(γ) gives $\dfrac{(a^2+ab+ab^2)\,(a-b)}{ab\,(a^2+ab-ab+b^2)} = \dfrac{(a+b+b^2)\,(a-b)}{b\,(a^2+b^2)}.$

(δ) gives

$$\frac{a+\sqrt{(a^2-x^2)}-a+\sqrt{(a^2-x^2)}}{a^2-a^2+x^2} = \frac{2\sqrt{(a^2-x^2)}}{x^2}.$$

(ϵ) gives

$$(\sqrt{a})^{\frac{2}{3}-\frac{1}{6}}+6\,\{a^3\,b\sqrt[3]{(a^3\,bc)}\}^4 = (\sqrt{a})^{\frac{1}{2}}+6\,a^{12}b^4a^4b^{\frac{4}{3}}c^{\frac{4}{3}}$$

$$= a^{\frac{1}{4}}+6\,a^{16}\,b^{\frac{16}{3}}\,c^{\frac{4}{3}}.$$

11. For the expansion $(1+x)^n$, see *Private Tutor*, vol. i. p. 24; for the middle term, &c. see *Private Tutor*, vol. i. pp. 199 and seq.

12. $(a\,x-x^2)^{-\frac{1}{3}} = (a\,x)^{-\frac{1}{3}}\left(1-\dfrac{x}{a}\right)^{-\frac{1}{3}}$

$$= (a\,x)^{-\frac{1}{3}}\left\{1+\tfrac{1}{3}.\frac{x}{a}+\frac{-\frac{1}{3}.\,(-\frac{1}{3}-1)}{2}\,\frac{x^2}{a^2}-\frac{-\frac{1}{3}\,(-\frac{1}{3}-1)\,(-\frac{1}{3}-2)}{2.\,3}\,\frac{x^3}{a^3}+\&c.\right\}$$

$$= (a\,x)^{-\frac{1}{3}}\left\{1+\tfrac{1}{3}.\frac{x}{a}+\frac{2}{9}\,\frac{x^2}{a^2}+\frac{14}{81}\,\frac{x^3}{a^3}+\&c.\right\}.$$

Generally the pth term of $(1+u)^n = \dfrac{n.\,(n-1)\ldots(n-p+2)}{1.\,2.\,3\ldots(p-1)}\,u^{p-1}$,

see *Private Tutor*, vol. i. p. 198. Let $u = -\frac{x}{a}$, and $n = -m$; then, when p is even, $\left(-\frac{x}{a}\right)^{p-1}$ is negative, and $-m(-m-1) \times \ldots (-m-p+2)$ is negative, $\therefore$ the pth term is positive; also, when p is odd, $\left(-\frac{x}{a}\right)^{p-1}$ is positive, and so also is $-m(-m-1) \times \ldots (-m-p+2)$, $\therefore$ the pth term is positive. Hence, in either case, the pth term of $\left(1-\frac{x}{a}\right)^{-m}$, is

$$\frac{m\,(m+1)\,(m+2)\ldots\ldots(m+p-2)}{1.\;\;2.\;\;3\;\;\ldots\ldots\;\;(p-1)}.\;\left(\frac{x}{a}\right)^{p-1}.$$

Let $m = -\frac{1}{3}$; then the pth term of $\left(1-\frac{x}{a}\right)^{-\frac{1}{3}}$ is

$$\frac{\frac{1}{3}.\,\frac{4}{3}.\,\frac{7}{3}.\,\frac{10}{3}\ldots\ldots\frac{1+3(p-2)}{3}}{1.\;\;2.\;\;3.\;\;4.\ldots\;\;(p-1)}.\;\left(\frac{x}{a}\right)^{p-1},$$

$$\text{or, } \frac{1}{3^{p-1}}.\;\frac{4.\,7.\,10.\ldots\{1+3(p-2)\}}{2.\,3.\,4.\;\ldots\;\;(p-1)}.\;\left(\frac{x}{a}\right)^{p-1},$$

whence the general term required is evident.

13. $\sqrt{\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}-2\right)} = \frac{a}{b}-\frac{b}{a}$ by common extraction.

By the rule for binomial surds,

$$\sqrt{(12\sqrt{-1}-5)} = \sqrt{(-5+12\sqrt{-1})},$$

$$=\sqrt{\frac{-5+\sqrt{(25+144)}}{2}}+\sqrt{\frac{-5-\sqrt{(25+144)}}{2}}$$

$$=\sqrt{\frac{-5+13}{2}}+\sqrt{\frac{-5-13}{2}}.$$

$$=2+3\sqrt{-1},$$

which is easily verified by involution.

14. The pth term is $\frac{n.\,(n-1)\,(n-2)\ldots(n-p+2)}{1.\;\;2.\;\;3\,\ldots\ldots\,(p-1)}\,a^{n-p+1}$

15. *Wood*, art. 90.

16. *Wood*, art. 377.

17. *Wood,* art. 182.

18. If a be the first term, b the common difference; then $2n$ terms $= \{2a + (2n-1)b\}n$, and n terms $= \{2a + (n-1)b\}\frac{n}{2}$; $\therefore$ the latter half of $2n$ terms $= \{2a + (2n-1)b\}n - \{2a + (n-1)b\}\frac{n}{2} = \{2a + (3n-1)b\}\frac{n}{2}$, which is the sum of $3n$ terms of the series.

19. $$\frac{\sqrt{\frac{4}{3}}\sqrt{a^3} - \sqrt{\frac{3}{4}}\sqrt{a^3}}{3\sqrt{\frac{3}{a}}} = \tfrac{1}{3}\sqrt{\tfrac{4}{9}\,a}\sqrt{a^3} - \tfrac{1}{3}\sqrt{\tfrac{1}{4}\,a}\sqrt{a^3}$$
$$= \frac{2a}{9}\sqrt{a} - \frac{a}{6}\sqrt{a} = \frac{a\sqrt{a}}{3}\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{a\sqrt{a}}{18}.$$

20. Effect $\propto$ cause $\times$ time; $\therefore$ if x be the time required, and c, c_1 the given causes, $a : a_1 :: c \times t : c_1 \times t_1$, and $a : c \times t :: b : (c + c_1) \times x$

But $c_1 = \dfrac{a_1 t}{a t_1} \times c$

$$\therefore x = \frac{b\,t}{a}\cdot\frac{c}{c + c_1} = \frac{b\,t}{a}\cdot\frac{c}{c + \frac{a_1 t}{a t_1}c} = \frac{b\,t\,t_1}{a\,t_1 + a_1 t}$$

21. The square being $c^2 = a^2 - b^2$, we have *Wood,* art. 258.
$$\sqrt{(a+b)} = \sqrt{\frac{a + \sqrt{(a^2 - b^2)}}{2}} + \sqrt{\frac{a - \sqrt{(a^2 - b^2)}}{2}}$$
$$= \sqrt{\frac{a+c}{2}} + \sqrt{\frac{a-c}{2}}.$$

22. If a be the digit, which recurs $2n$ times, the number is
$$\begin{aligned} N &= a + a.\,10 + a.\,10^2 + \ldots\ldots a.\,10^{2n-1} \\ &= a\,\{1 + 10 + (11-1)^2 + (11-1)^3 + \ldots (11-1)^{2n-1}\} \\ &= a\,\{11 \times Q + 1 - 1 + 1 - 1 \ldots \text{ to } (2n-2) \text{ terms}\} \\ &= 11\,a.\,Q, \end{aligned}$$
which is divisible by 11.

This is only a particular case of the general proposition, that *if the sum of the digits in the odd places of any number = sum of those in the even places, the number is divisible by* 11. See *Wright's* Self-Instructions in Arithmetic, p. 48.

23. The operation is $\dot{1}6\dot{0}.6\dot{1}$ (12. 86

$$
\begin{array}{rr}
 & 1 \\
\hline
22) & 60 \\
 & 44 \\
\hline
248) & 1861 \\
 & 1714 \\
\hline
2546) & 14900 \\
 & 12830 \\
\hline
 & 2090 \\
\hline
\end{array}
$$

Proof by 11

$$
\begin{array}{ccc}
 & 3 & \\
6 & \times & 6 \\
 & 3 &
\end{array}
\quad \text{gives } 30
$$

See *Barlow*, T.N.p.234.

2090 remainder.

24. (α) Since $y = \frac{b}{c}x$, $\therefore\ x^3 - \frac{b^3}{c^3}x^3 = d$

$$\therefore\ x = c\left(\frac{d}{c^3 - b^3}\right)^{\frac{1}{3}} \text{ and } y = b\left(\frac{d}{c^3 - b^3}\right)^{\frac{1}{3}}$$

each of which has 3 roots. See *P. T.* vol. i. p. 399.

(β) Clearing of fractions.

$$\sqrt{(1+x)} - \sqrt{(1-x^2)} = \sqrt{(1- \quad} + \sqrt{(1-x^2)}$$

$$\therefore \sqrt{(1+x)} - \sqrt{(1-x)} = 2\sqrt{(1-x^2)}$$

$$\therefore\ 1 + x + 1 - x - 2\sqrt{(1-x^2)} = 4 - 4x^2$$

or $\sqrt{(1-x^2)} = 2x^2 - 1$

$$\therefore\ -x^2 = 4x^4 - 4x^2$$

$$\therefore\ x^4 - \frac{3x^2}{4} = 0$$

$\therefore\ x = 0$ and $x^2 = \frac{3}{4}$, or the roots are $0, 0, \frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{2}$.

(γ) Rationalizing the denominator by multiplying by $ax + 1 + \sqrt{(a^2x^2 - 1)}$, and we get

$$\frac{\{ax + 1 + \sqrt{(a^2x^2 - 1)}\}^2}{2(ax + 1)} = \frac{b^2x}{2}$$

$$\text{or } \frac{(ax+1)\ \{\sqrt{(ax+1)} + \sqrt{(ax-1)}\}^2}{ax+1} = b^2x$$

$$\therefore\ 2ax + 2\sqrt{(a^2x^2 - 1)} = b^2x$$

$$\therefore\ 4a^2x^2 - 4 = b^4x^2 + 4a^2x^2 - 4ab^2x^2$$

$$\therefore\ (4a - b^2)\,b^2x^2 = 4$$

$$\text{and } x = \pm\frac{2}{b\sqrt{(4a - b^2)}}.$$

TRINITY COLLEGE, 1826.

1. Proper, improper, decimal, terminating decimal, circulating decimal, non-terminating decimal, and continued fractions.

2. The sum is $15 + \frac{1}{2} + \frac{3}{10} = 15\frac{4}{5}$.

3. $\frac{2}{7} : \frac{3}{4} :: \frac{5}{6}$: fourth proportional $= \frac{5}{8} \times \frac{7}{2} = \frac{35}{16} = 2\frac{3}{16}$

4. The root is 17. 2.

5. For $12.1 = 12 + \frac{1}{10} = \frac{121}{10} = \frac{11^2}{10}$ no square.

6. *Wood*, art. 79.

7. $a^{\frac{3}{2}}$ means the square root of the cube of a, and a^{-5} means $\frac{1}{a^5}$ or 1 divided by the 5th power of a. The original meaning of a^m is the continued product of $a \times a \times a \ldots . m$ factors, $\therefore a^{-5}$ and $a^{\frac{3}{2}}$ are not included in this.

8.

$$(a^2 + x)^{\frac{1}{2}} = a\left(1 + \frac{x}{a^2}\right)^{\frac{1}{2}} = a\left(1 + \frac{1}{2}\frac{x}{a^2} + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2}\frac{n^2}{a^4} + \&c.\right)$$

$$= a\left(1 + \frac{1}{2}\frac{x}{a^2}\right) \text{ nearly}$$

$$= a + \frac{x}{2a} \text{ nearly.}$$

Hence $(145)^{\frac{1}{2}} = (12^2 + 1)^{\frac{1}{2}} = 12 + \frac{1}{24}$ nearly.

9. (1) $x = 2$

(2) $9 - 6x = 12 - 8x + 10 \therefore x = \frac{13}{2}$

(3) $x^2 - 2x = 3, \therefore x = 1 \pm \sqrt{4} = 3$ and -1

(4) $x^2 + 4x + 4 = 4x + 5 \therefore x = \pm 1$

(5) $\left.\begin{aligned} x^2 - 2xy + y^2 &= \frac{121}{4} \\ 4xy &= 80 \end{aligned}\right\} \therefore x + y = \sqrt{\frac{441}{4}} = \pm\frac{21}{2}$

and $x - y = \frac{11}{2}$

$\therefore\ x = 8$ and $-\frac{5}{2}$ and $y = \frac{5}{2}$ and -80.

(6) Multiply the 2ᵈ by 3, and subtract the result from the first, and we get

$$11x - 25y = -17,$$

also, adding the 3ᵈ to the second,

$$9y - 2x = 12.$$

From these equations we get $x = 3$, $y = 2$ and $z = 4$.

10. Let x and y be the two parts; then by the question

$$x^2 + y = y^2 + x$$

$$\therefore\ x^2 - y^2 = x - y$$

$$\therefore\ x + y = 1 \text{ the number required.}$$

11. The cost $= \frac{100}{3} + 50$ pence, and they make $200 \times \frac{2}{5} = 80$ pence,

$$\therefore \text{ the loss} = 3\tfrac{1}{4}d.\ \tfrac{1}{3}.$$

12. The cask contains $50 \times 9 = 450$ bottles, and has leaked $3\frac{1}{2} \times 9 = \dfrac{63}{2}$ bottles;

and $\therefore$ the wine to be sold $= 450 - \dfrac{63}{2} = \dfrac{737}{2}$ bottles.

But the gain is to be $37l.\ 1 \times \dfrac{15}{100} = (37.\ 1) \times \dfrac{3}{20}$

$$= \frac{111l. 3s.}{20}\ \frac{2223}{400}l.$$

$\therefore$ the price, per bottle, is

$$\left(\frac{741}{20} + \frac{2223}{400}\right)\frac{2}{737} = \left(741 + \frac{2223}{20}\right)\frac{1}{7370} = \frac{17043}{147400}$$

and the price, per dozen, $= \dfrac{17043 \times 12}{147400} = \dfrac{17043 \times 3}{36850}$

$$= \frac{51129}{36850} = 1\,\frac{14279}{36850}l.$$

$$= 1l.\ \frac{28558}{3685}s. = 1l.\ 7s.\ 8\tfrac{3}{4}d.\ \frac{3649}{3685}.$$

13. Let a be the first term, b the common difference, and x the number of terms required; then, by the question,

$$a + a + b = 18,\ a + 2b = 12$$
$$\therefore\ 3a + 3b = 30 \text{ and } a+b = 10$$
$$\text{whence } a = 8 \text{ and } b = 2.$$
$$\therefore\ \{2a + (x - 1)b\}\frac{x}{2} = 28,$$

$$\text{or } (16 + 2x - 2)\frac{x}{2} = 28, \text{ or } x^2 + 7x = 28$$

$\therefore$ solving the quadratic

$$x = -\frac{7}{2} \pm \sqrt{\frac{49 + 112}{4}}$$
$$= \frac{-7 \pm \sqrt{161}}{2}$$

which shows the question to be imperfectly stated.

14. The series is geometric, the ratio being $\frac{4}{10}$ or $\frac{2}{5}$, and the sum is $\frac{4 \times 1031 \times 3157}{78125}$, or $166\,\frac{50718}{78125}$.

15. $\sqrt{(28 - 6\sqrt{3})} = \sqrt{\frac{28 + \sqrt{(28^2 - 6^2 \times 3)}}{2}}$
$$- \sqrt{\frac{28 - \sqrt{(28^2 - 6^2 \times 3)}}{2}}$$
$$= \sqrt{\frac{28+24}{2}} - \sqrt{\frac{28-24}{2}} = \sqrt{26} - \sqrt{2}. \text{ See } \textit{Wood}, \text{ art. } 258.$$
$$(2\sqrt{-1}) = \sqrt{(0+2\sqrt{-1})} = \sqrt{\frac{0+4}{2}} + \sqrt{\frac{0-4}{2}} = \sqrt{2} + \sqrt{2}.\ \sqrt{-1}.$$

16. *Wood*, art. 255.

17. See *Private Tutor*, vol. i. p. 21.

18. The general or pth term of $(c + x)^m$ is
$$\frac{n(n-1)\ .\ .\ .\ (n-p+2)}{1.\ 2\ \ .\ .\ .\ (p-1)}\, c^{n-p+1}\, x^{p-1}$$
Here $c = a^2$, $x = -b^2$, $n = \frac{3}{2}$, $p = 5$,

$\therefore$ the 5th term is

$$\tfrac{3}{2}.\frac{(\tfrac{3}{2}-1)\,(\tfrac{3}{2}-2)\,(\tfrac{3}{2}-3)}{1.\quad 2.\quad 3.\quad 4}(a^2)^{\frac{3}{2}-4}(-b^2)^4$$

$$\text{or, } \frac{3.\ 1.\ 1.\ 3}{16 \times 2.\ 3.\ 4} a^{-5}\, b^8$$

$$\text{or, } \frac{3}{128} \times \frac{b^8}{a^5}.$$

19. *Barlow's* Theory of Numbers, art. 12.

20. When a and b are not prime to each other, and c does not contain their common measure. *Barlow*, art. 160.

21. Let x be the number of crowns, y those of the seven-shilling pieces, which, together, = 13 Napoleon's; then

$$5\,x + 7\,y = 13 \times 16 = 208.$$

Generally, if $a\,x + b\,y = c$

and $a\,q - b\,p = 1$,

then the number of positive solutions of $ax + b\,y = c$ is the greatest integer in $\frac{cq}{b} - \frac{cp}{a}$, q and p being the least values which satisfy $a\,q - bp = 1$. *Barlow*, art. 161.

Now, $5\,q - 7\,p = 1$ gives $q = 3$, $p = 2$,

$$\therefore\ \frac{cq}{b} - \frac{cp}{a} = 208 \times \left(\frac{3}{7} - \frac{2}{5}\right) = \frac{208}{35} \text{ gives } 6.$$

Hence the number of ways required is 5. The ways of payment are easily found to be

$$y = 4, 9, 14, 19, 24, 29$$
$$x = 36, 29, 22, 15, 8, 1.$$

22. *Wood*, art. 380. The common difference of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, being $\frac{1}{6}$, the series continued is

$$\ldots\ \tfrac{1}{2}+\tfrac{3}{6},\ \tfrac{1}{2}+\tfrac{2}{6}\ \tfrac{1}{2}+\tfrac{1}{6},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{6},\ \tfrac{1}{6}-\tfrac{1}{6},\ \tfrac{1}{6}-\tfrac{2}{6},\ \tfrac{1}{6}-\tfrac{3}{6},\ \tfrac{1}{6}-\tfrac{4}{6}\ \ldots$$

$$\text{or } \ldots\ 1,\ \tfrac{5}{6},\ \tfrac{2}{3},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{6},\ 0,\ -\tfrac{1}{6},\ -\tfrac{1}{3},\ -\tfrac{1}{2},\ -\ \ldots$$

$\therefore$ the harmonic series is

$$\ldots\ 1,\ \tfrac{6}{5},\ \tfrac{3}{2},\ 2, 3, 6, \infty,\ -6,\ -3,\ -2\ \ldots$$

23. Present value for 19 years, beginning now,

$$= \frac{19\,\text{A} + 19 \times 9\,r\,\text{A}}{1 + 19\,r}\ \textit{Wood}, \text{ art. } 394.$$

$$= \frac{19 \times 50\left(1 + \frac{9}{20}\right)}{1 + \frac{19}{20}} = \frac{19 \times 50 \times 29}{39}$$

P.V. for 7, beginning now, $= \frac{7 \times 50\left(1 + \frac{3}{20}\right)}{1 + \frac{7}{20}} = \frac{7 \times 50 \times 23}{27}$

$$\therefore \text{ P. V. required } = \frac{50}{3}\left(\frac{19 \times 29}{13} - \frac{7 \times 23}{9}\right),$$

which is easily reduced.

24. *Bonnycastle's* Arithmetic, p. 149.

25. Divide 4321 by 7, successively, in the quinary scale, and the remainders will be the digits in the septenary scale. *Barlow.* *Ans.* 1460.

26. A number is divisible by 3 when the sum of its digits is; by 8, when the number denoted by its 3 last digits is; by 11, when sum of the digits in odd places=sum in the even.

27. When the denominator is of the form $2^m \times 5^n$.

28. See *Woodhouse's* Trig. Log 3 is obtained from

$$\log(a+b), = \log a + \frac{b}{a+b} + \tfrac{1}{3}\left(\frac{b}{a+b}\right)^3 + \tfrac{1}{5}\left(\frac{b}{a+b}\right)^5 + \&c.$$

TRINITY COLLEGE, 1827.

1. $3\frac{1}{3} = \frac{10}{3}$ and $\frac{2}{7}$ of $9\frac{2}{5} = \frac{2}{7} \times \frac{47}{5} = \frac{94}{35}$,

$\therefore$ they are as $\frac{15}{4}, \frac{10}{3}, \frac{94}{35}$, or as $\frac{1575}{420}, \frac{1400}{420}, \frac{1128}{420}$,

or as 1575, 1400, 1128.

2. $\frac{7}{18}$ of a guinea $= \frac{7}{18} \times 21s. = 8s.\ 2d.$

Also, $\frac{2}{5}$ of $2s.\ 4\frac{1}{2}d. = \frac{4s.\ 5d.}{5} = \frac{53}{60}s.$

$\therefore$ fraction required is $\dfrac{53}{60} \times \dfrac{1}{2\frac{1}{2}} = \dfrac{53}{150}$.

3. *Wood.* The quotient is 5000.

4. 3*s*. 4*d*. $= \frac{1}{6}$ of a pound $= \frac{1}{3} \times .5l. = .16666\ldots$
and $\frac{1}{2} = \frac{1}{80}$ of 3*s*. 4*d*. $= .00208\ldots$
$\therefore$ the decimal is $.16874\ldots.$

5. *Wood.* The greatest common measure of 13536 and 23148, is 4.

The algebraic expressions have no greatest common measure except unit. This may be seen by the common method, and also from the consideration that $x^4-(p-q)\,x^3+(p-q)\times q^2\,x-q^4 = x^4-q^4-(p-q)\,x\,(x^2-q^2)$

$$= (x-q)\,(x+q)\,\{x^2-(p-q)\,x+q^2\},$$

none of the factors of which is a factor of the other expression.

6.

(1) $\dfrac{x^3+a^3}{a+x} = \dfrac{x^3+a^3}{x+a} = x^2 - ax + a^2$.

(2) $\dfrac{(a+b)^2}{a^2-b^2} - \dfrac{(a^2-b^2)^2-(a^2+b^2)^2}{a^4-b^4} = \dfrac{a^2+2ab+b^2}{a^2-b^2} + \dfrac{4a^2b^2}{a^4-b^4},$

$$= \frac{a^4+2a^3b+6a^2b^2+2ab^3+b^4}{a^4-b^4}.$$

(3) $\dfrac{a^2+2ab\sqrt{-1}-b^2+a^2-2ab\sqrt{-1}-b^2}{a^2+b^2} = \dfrac{2(a^2-b^2)}{a^2+b^2}$.

(4) By Evolution, the root is $\dfrac{3y}{4x} + \dfrac{2}{7}\dfrac{x}{y} - 5$.

7. *Wood*, art. 111., and seq.

8. (1) $x = \dfrac{b^2}{a-c}$.

(2) The denominator of the right hand side of the equation being (9), (it has been dropt in printing); we have

$$\left.\begin{aligned} 126y - 9x + 54y - 9 &= 7x - 21 \\ 7x - 35y + 56 &= 27x - 117y \end{aligned}\right\}$$

$$\text{or, } \left.\begin{aligned} 4x - 45y &= 3 \\ 10x - 41y &= 28 \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore\ 20x - 225y &= 15 \\ 20x - 81y &= 56 \end{aligned}\right\}.$$

$$144\,y = 41 \quad \therefore\ y = \frac{41}{144};\ \text{whence } x.$$

(3) $x - 2 - \dfrac{x^3 - 8}{x^2 + 5} = 0$

$$\therefore\ \text{dividing by } x - 2$$

$$1 - \frac{x^2 + 2\,x + 4}{x^2 + 5} = 0$$

$$\therefore\ x = 2 \text{ and } \tfrac{1}{2}.$$

(4) This equation is erroneously printed. It ought to have been

$$2\,x^2 - 2\,x + 2\,\sqrt{(2\,x^2 - 7\,x + 6)} = 5\,x - 6.$$

It thus becomes

$$2\,x^2 - 7\,x + 6 + 2\sqrt{(2\,x^2 - 7\,x + 6)} = 0$$

$$\therefore\ \sqrt{(2\,x^2 - 7\,x + 6)}\ \{\sqrt{(2\,x^2 - 7\,x + 6)} + 2\} = 0$$

$$\therefore\ 2\,x^2 - 7\,x + 6 = 0, \text{ and } \sqrt{(2\,x^2 - 7\,x + 6)} + 2 = 0.$$

Hence $x^2 - \dfrac{7}{2}\,x = -\,3$, and $x^2 - \dfrac{7}{2}\,x = -\,1$

$$\therefore\ x - \frac{7}{4} = \pm\sqrt{\left(\frac{49}{16} - 3\right)} \text{ and } x - \frac{7}{4} = \pm\sqrt{\left(\frac{49}{16} - 1\right)}$$

$$\text{or, } x = \frac{7 \pm 1}{4} \text{ and } x = \frac{7 \pm \sqrt{33}}{4},$$

or the values of x are $2,\ \dfrac{3}{2},\ \dfrac{7 + \sqrt{33}}{4},\ \dfrac{7 - \sqrt{33}}{4}$.

(5) Square the second.

$$x^2 + y^2 = 4 + 2\,xy$$

$$\therefore\ x^4 + y^4 = 16 + 16\,xy + 2\,x^2\,y^2 = 272$$

$$\therefore\ x^2\,y^2 + 8\,x\,y = 128$$

$$\therefore\ x\,y = 8 \text{ and } -\,16$$

$$\left.\begin{aligned} \therefore\ x + y &= \pm\,6 \text{ and } \pm\,2\sqrt{-\,15} \\ \text{But } x - y &= 2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore\ x &= 4,\ -\,2,\ 1 + \sqrt{-\,15},\ 1 - \sqrt{-\,15} \\ y &= 2,\ -\,4,\ -\,1 + \sqrt{-\,15},\ -\,1 - \sqrt{-\,15} \end{aligned}\right\}$$

(6) $2\,x^3 - 2\,x^2 = 1 - x^2 = -\,(x^2 - 1)$

$$2\,x^2\,(x - 1) + x^2 - 1 = 0$$

$$\therefore\ (x - 1)\,(2\,x^2 + x + 1) = 0$$

Hence, the roots are 1, $\dfrac{-1+\sqrt{-7}}{4}$, $\dfrac{-1+\sqrt{-7}}{4}$.

(7) Making $w = \dfrac{4y-2}{9}$, $w' = \dfrac{w+2}{4}$ by the usual method we shall get

$$w = 4w' - 2,$$
$$y = \frac{9w+2}{4},$$
$$x = \frac{200-13y}{9}.$$

Then make $w' = 0, 1, 2, 3, 4, \&c.$
$$\therefore w = -2, 2, 6, 10, 14, \&c.$$
$$y = -4, 5, 14, 23, 32, \&c.$$
and $x = 15, 2, \&c.$

so that the only positive integer corresponding values of x and y are
$$x = 15, 2,$$
$$y = 5, 14.$$

(8) Since $x - y = \dfrac{144}{x+y}$; $\therefore x + y$ must be some of the divisors of 144, that is,

$$x + y = 1, 2, 4, 8, 16, 3, 9, 6, 12, 24, 48, 18, 36, 72, 144,$$
$$\therefore x - y = 144, 72, 36, 18, 9, 28, 16, 24, 12, 6, 3, 8, 4, 2, 1.$$

Hence, the positive integer values of x and y are
$$x = 12, 15, 13, 20, 37,$$
$$y = 0, 9, 5, 16, 35.$$

9. The Arithmetic mean $= \dfrac{a+b}{2} = \text{A}$ suppose,

Geometric $= \sqrt{(ab)} = \text{G}$,

Harmonic $= \dfrac{2ab}{a+b} = \text{H}$,

$$\text{A} : \text{G} :: \text{G} : \frac{\text{G}^2}{\text{A}} = \frac{2ab}{a+b} = \text{H}.$$

The Arithmetic mean is the greatest.

10. (1) $a : b :: ma : mb :: nc : nd :: qe : qf$,
$\therefore ma : nc :: mb : nd$,

$\therefore\ ma : ma+nc :: mb : mb+nd,$

$\therefore\ qe : qf :: ma : mb :: ma+nc : mb+nd,$

$\therefore\ qe : ma+nc :: qf : mb+nd,$

$\therefore\ qe : ma+nc+qe :: qf : mb+nd+qf,$

$\therefore\ a : b :: qe : qf :: ma+ne+qe : mb+nb+qf,$

and so on.

(2) $a^m : b^m :: c^m : d^m,$

$$\therefore\ a^m+d^m = a^m + \left(\frac{b\,c}{a}\right)^m = a^m + \left(\frac{b}{a}\right)^m .\ c^m,$$

which is $> b^m + c^m$ on two accounts, because a is $> b$, and $\therefore\ \frac{b}{a}.\ c$ is $> c$.

(3) Here is a misprint. It ought to be,

"If $a+b : c+d :: c-d : a-b$."

Then $a^2-b^2 = c^2-d^2$,

$\therefore\ a^2-c^2 = b^2-d^2$, or $(a+c)\ (a-c) = (b+d)\ (b-d)$,

$\therefore\ a+c : b+d :: b-d. : a-c.$

11. (α) Here the common difference is 3,

$$\therefore\ \text{sum} = (4+10\times 2)\ \frac{11}{2} = 132.$$

(β) This series is geometric, its ratio being $\frac{3}{2}$;

$$\therefore\ \text{S} = \frac{a\,r^n - a}{r-1} = \frac{3\left\{\left(\frac{3}{2}\right)^5 - 1\right\}}{\frac{3}{2} - 1} = \frac{3\,(3^5-2^5)}{2^4} = \frac{633}{16}.$$

(γ) Ought to be $-5-3-1\ \ldots\ldots$;

then the sum is 16.

The value of

$$.012363636\ldots = \frac{12}{1000} + \frac{36}{100000} + \frac{36}{10000000} + \&c.$$

$$= \frac{12}{1000} + \frac{36}{100000}\left(1 + \frac{1}{100} + \frac{1}{10^4} + \&c.\right)$$

$$= \frac{12}{1000} + \frac{36}{100000}.\ \frac{1}{1-\frac{1}{100}}\quad \textit{Wood}, \text{ art. } 224.$$

$$= \frac{12}{1000} + \frac{3600}{100000} \cdot \frac{1}{99} = \frac{12}{1000} + \frac{4}{1000} \cdot \frac{1}{11} = \frac{3}{250} + \frac{1}{11 \times 250}$$

$$= \frac{34}{11 \times 250} = \frac{17}{11 \times 125} = \frac{17}{1375}.$$

12. See *Private Tutor*, vol. i. p. 21.

13. See *Private Tutor*, vol. i. p. 263.

14. Let $2m+1$, $2n+1$ be the two odd numbers; then

$$(2m+1)^2 - (2n+1)^2 = 4(m^2 - n^2) + 4(m - n)$$
$$= 4(m - n)(m + n + 1),$$

in which, if m and n be both odd or both even, $m - n$ is divisible by 2; if m and n be one odd and the other even, then $m + n + 1$ is even; so that $(m - n)(m + n + 1)$ is divisible by 2.

$\therefore (2m + 1)^2 - (2n + 1)^2$ is divisible by 4×2, or 8. Every prime number > 5 is contained in the expression $6n + 1$. Consequently the difference of the squares must be of the form

$$(6m + 1)^2 - (6n + 1)^2 =$$

$36(m^2 - n^2) + 12(m - n) = 12(m - n)\{3(m + n) + 1\}$, which, by reasoning like the above, is shown to be divisible by 24.

15. Let $N = a + br + cr^2 + dr^3 + \ldots$

$$= a + b(r + 1 - 1) + c(r + 1 - 1)^2 + \&c.$$
$$= a - b + c - d + e \ldots + (r+1)\{b + (r+1)c - 2c + \&c.\}$$
$$= (r + 1)\{b + (r + 1)c - 2c + \&c.\} \text{ by hypothesis,}$$

which is divisible by $(r + 1)$.

16. Transform the given number (N) from the common scale of notation to that in which the radix of the system is n, and the new digits will be the numbers of the weights to be taken. *Ex.* Required to find how many of the weights, $1, 5, 5^2, 5^3 \ldots$ will weigh accurately 1769lbs.

$$\begin{array}{r} 5\,)\,1\,7\,6\,9 \\ \hline 5\,)\,3\,5\,3-4 \\ \hline 5\,)\,7\,0-3 \\ \hline 5\,)\,1\,4-0 \\ \hline 2-4 \end{array}$$

$\therefore$ in the new scale the number is 24034,

$\therefore$ the weights are 4 of 1 lb. 3 of 5 lbs. 4 of 125 lbs. and 2 of 625 lbs.

17. The only true principle on which this estimate can be made is that of *Malcolm*, viz., that if the equated time (x) be between t_m and t_{m+1}, then the aggregate interest of the sums $P_1, P_2 \ldots\ldots P_m$ for the intervals $x - t_1$, $x - t_2 \ldots\ldots x - t_m$ must $=$ the aggregate discount of the sums $P_{m+1}, P_{m+2}\ldots\ldots$ for the intervals $t_{m+1} - x$, $t_{m+2} - x$, &c.

Let R be the amount of 1*l.* for a year, compound interest. Then the amount of P_1 for $(x - t_1)$ years is $P_1 R^{x-t_1}$, (*Wood*, art. 397), and similarly for the other sums. Hence the aggregate interest is

$$P_1R^{x-t_1}-P_1+P_2R^{x-t_2}-P_2+P_3R^{x-t_3}-P_3\ldots P_mR^{x-t_m}-P_m,$$

$$\text{or,}\left(\frac{P_1}{R^{t_1}}+\frac{P_2}{R^{t_2}}+\ldots\ldots\frac{P_m}{R^{t_m}}\right)R^x-(P_1+P_2+P_3+\ldots P_m),\ldots(a)$$

Again, the discount of any sum $=$ worth when due $-$ its present worth.

Let w be the present worth of s pounds due n years hence at compound interest; then $w\,R^n = s$, $\therefore w = \frac{s}{R^n}$, $\therefore$ the discount of s, at compound interest, $= s - \frac{s}{R^n}$. Hence the above described aggregate discount is

$$P_{m+1} - \frac{P_{m+1}}{R^{\overset{t-x}{m+1}}} + P_{m+2} - \frac{P_{m+2}}{R^{\overset{t-x}{m+2}}} + \text{\&c.}$$

$$\text{or, } P_{m+1} + P_{m+2} + P_{m+3} + \dots$$
$$-\left(\frac{P_{m+1}}{R^{t_{m+1}}} + \frac{P_{m+2}}{R^{t_{m+2}}} + \&c.\right)R^x \dots\dots (b)$$

which being equal to the aggregate of interests (a), we have

$$R^x = \frac{P_1 + P_2 + \dots P_m + P_{m+1} + \dots}{\frac{P_1}{R^{t_1}} + \frac{P_2}{R^{t_2}} + \dots \frac{P_m}{R^{t_m}} + \frac{P_{m+1}}{R^{t_{m+1}}} + \dots\dots}$$

whence x may be found by the Tables of Logarithms.

Ex. *Supposing* 100*l.* *to be payable* 1 *year hence, and* 105*l.* 3 *years hence; what is the equated time to pay the whole, allowing compound interest at* 5 *per cent. per annum.*

Here $P_1 = 100$, $P_2 = 105$,

$$t_1 = 1,\ t_2 = 3, \text{ and } R = 1 + \frac{1}{20} = \frac{21}{20},$$

$$\therefore \left(\frac{21}{20}\right)^x = \frac{205}{100 \times \frac{20}{21} + 105 \times \frac{20^3}{21^3}} = \frac{41 \times 21^3}{400 \times 21^2 + 21 \times 20^3}$$

$$= \frac{41 \times 21^2}{20^2 \times (21 + 20)} = \left(\frac{21}{20}\right)^2$$

$$\therefore x = 2,$$

which is evidently true; for the interest of 100*l.* for 1 year $= 5l.$ and the present worth of 105*l.* due a year hence, $= \frac{105}{R}$ $= 105 \times \frac{20}{21} = 100l.$ and $\therefore$ the discount is 5*l.*

Ex. 2. *What is the equated time of* 400*l.* *due* 2 *years hence and* 2100 *due* 8 *years hence at* 5 *per cent. compound interest?*

Here

$$\left(\frac{21}{20}\right)^x = \frac{2500}{400 \times \frac{20^2}{21^2} + 2100 \times \frac{20^8}{21^8}} = \frac{25}{4 \times \frac{20^2}{21^2} + \frac{20^8}{21^7}}$$

$$= \frac{25 \times 21^7}{20^2(4 \times 21^5 + 20^6)} = \frac{21^7}{4^3(21^5 + 5 \times 20^5)} \text{ whence } x \text{ may}$$

be found by reference to the tables. Or thus

$$\left(\frac{21}{20}\right)^x = \frac{1801088541}{1285382464}$$

$$\text{But } \left(\frac{21}{20}\right)^x = \left(1 + \frac{1}{20}\right)^x = 1 + \frac{x}{20} \text{ nearly}$$

$$\therefore \frac{x}{20} = \frac{515706077}{1285382464}$$

$$\therefore \ x = \frac{2578530385}{321345616} = 7 \text{ nearly.}$$

18. $\sqrt{(1 + \frac{1}{3}\sqrt{5})} = \frac{1}{\sqrt{3}}\sqrt{(3+\sqrt{5})} = \sqrt{\frac{5}{6}} + \sqrt{\frac{1}{6}}$

$$= \tfrac{1}{6}(\sqrt{30} + \sqrt{6})$$

$\sqrt[4]{(-16a^4)} = 2a\sqrt{(0+\sqrt{-1})} = 2a\left\{\sqrt{\frac{0+\sqrt{(0^2+1)}}{2}} + \sqrt{\frac{0-\sqrt{(0^2+1)}}{2}}\right\}$

$$= 2a\,(\sqrt{\tfrac{1}{2}} + \sqrt{-\tfrac{1}{2}}) = a\sqrt{2}\,(1. + \sqrt{-1}).$$

19. $\text{Log } 90 = \log 9 + \log 10 = \log 9 + 1 = \log (6 \times 15),$

$$= \log 6 + \log 15 = 1.9542426,$$

$$\therefore \log 9 = .9542426.$$

Hence, $\log 3 = \frac{1}{2}\log 9$ is known,

$\therefore \log 2 = \log 6 - \log 3$ is known,

and $\therefore \log 8 = 3 \log 2$ is known.

20. The probability that an event will happen

$$= \frac{\text{number of favorable cases}}{\text{total number of cases}}.$$

The number of permutations of 52 cards $= 52.\ 51.\ \ldots\ 2.\ 1,$

and that of 13 cards $= 13.\ 12.\ \ldots\ 2.\ 1,$

$$\therefore \text{ total number of deals} = \frac{52.\ 51.\ \ldots\ 2.\ 1}{(13.\ 12.\ \ldots\ 2.\ 1)^4}.$$

The permutations of 48 cards $= 48.\ 47.\ \ldots\ 2.\ 1,$

and of 12 . . . . $= 12.\ 11.\ \ldots\ 2.\ 1,$

$\therefore$ total number of ways in which 48 cards can be dealt, is

$$\frac{48.\ 47.\ \ldots\ 2.\ 1}{(12.\ 11.\ \ldots\ 2.\ 1)^4},$$

and each of these arrangements may be combined with the permutations of the four honors. Consequently, the number of cases favorable to the event in question is

$$\frac{48.47.\ldots.2.1}{(12.11.\ldots.2.1)^4} \times 4.3.2.1,$$

$$\therefore \text{ the probability required} = \frac{\dfrac{48.47.\ldots.2.1}{(12.11.\ldots.2.1)^4} \times 24}{\dfrac{52.51.\ldots.2.1}{(13.12.\ldots.2.1)^4}},$$

$$= \frac{13^4}{52.51.50.49} \times 24,$$

$$= \frac{13^3}{51.50.49} \times 6 = \frac{13^3}{17.25.49},$$

$$= \frac{2197}{20825} = \frac{1}{9+\frac{1}{2}}, \text{ nearly},$$

$$= \frac{2}{19}, \text{ nearly}.$$

Hence, it is 8 to 1, or 17 to 2, against the event.

TRINITY COLLEGE, 1828.

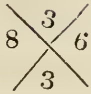
1. *Barlow's* Theory of Numbers, pp. 220 and seq.

2. Prop. 1. Book II. Euc.

3. Rule extended. *Place the multiplier under the multiplicand, units under units, tens under tens, tenths under tenths, &c. Multiply the multiplicand by the units' digit of the multiplier; then by the tens' digit, placing this latter result one place to the left of the former result; then by the hundreds' digit, placing the result one place further to the left, and so on. Again, multiply by the tenths' digit, placing the result one place to the right of the multiplicand; then multiply by the hundredths' digit, placing the result one place further to the right, and so on throughout. Then add up the columns, and the resulting sum will be the required product.*

EXAMPLE:

```
   9876 . 329
    453 . 7689
-----------------
  29628 . 987
 493816 . 45
3950531 . 6
   6913 . 4303
    592 . 57974
     79 . 010632
      8 . 8886961
-----------------
4481570 . 9463681.
-----------------
```



4481570 . 9463681. The same as by the common method.

4. The least common multiple of the denominators being 12, the sum is $\dfrac{8+9+10}{12} = \frac{27}{12} = \frac{9}{4} = 2\frac{1}{4}$.

5. $(18+\frac{1}{3}) \div (2+\frac{1}{4}+\frac{1}{24}) = \frac{55}{3} \div \frac{55}{24} = \frac{24}{3} = 8$.

6. His debts amount to 560. He pays 4*s.* in the pound. They receive 50*l.*, 30*l.*, 42*l.*

7. The root is 23. 12. For rule, see *Wood.*

8. $\dfrac{a^2-b^2}{5b} \times \dfrac{15a^2}{a+b} = \dfrac{3a^2}{b}(a-b)$.

9. $\dfrac{a^2+x^2}{a^2-x^2} = 1 + \dfrac{2x^2}{a^2} + \dfrac{2x^4}{a^4} + \dfrac{2x^6}{a^6} + \text{\&c. to } \infty$.

10. (1) $7x = 35$, $\therefore x = 5$.

(2) $36 - 9x - 12 = 12x - 48$,

$\therefore x = \frac{24}{7} = 3\frac{3}{7}$.

(3) $x = \dfrac{a+2c}{b-1}$.

(4) $$\left.\begin{aligned}2x + 2y &= 17\\ 10x - 26y &= 67\end{aligned}\right\},$$
$$\therefore\ x = 8,$$
$$y = \tfrac{1}{2}.$$

(5) $$x^2 - \frac{x}{3} = 34,$$

$$\therefore\ x = \tfrac{1}{6} \pm \sqrt{(\tfrac{1}{36} + 34)} = \frac{1 \pm 35}{6} = 6 \text{ and } -\tfrac{17}{3}.$$

(6) $$x^2 - 12x = -40,$$
$$\therefore\ x = 6 \pm \sqrt{-4} = 6 \pm 2\sqrt{-1}.$$

11. Let x be the shillings he had at first. Then,
$\frac{4}{5}x + 10 =$ what he had after the first sitting,
$$\therefore\ \tfrac{2}{3}(\tfrac{4}{5}x + 10) + 3 = 63,$$
$$\therefore\ x = 100s. = 5l.$$

12. Let x be the number bought. Then,
$$\frac{57}{x} = \text{prime cost per head.}$$
$$\therefore\ (x-8)\left(\frac{57}{x} + \frac{2}{5}\right) = \text{what he sold the remainder for} = 57l.$$
$$\therefore\ 2x^2 - 16x = 2280,$$
$$\therefore\ x^2 - 8x = 1140,$$
$$\therefore\ x = 4 \pm \sqrt{1156} = 4 \pm 34,$$
$$= 38.$$

13. *Wood*, art. 162.

14. See p. 47.

15. $x : y :: a^2 : b^2$, and $a : b :: \sqrt{(a + x)} : \sqrt{(a - y)}$,
$$\therefore\ x : y :: a + x : a - y,$$
$$\therefore\ x : a :: y : a - 2y,$$
$$\therefore\ 2x : a :: 2y : a - 2y.$$
But $x - y : y :: x + y : a - y$,
$$\therefore\ a - y = y\frac{x + y}{x - y},$$

$$\text{and } a - 2y = \frac{2y^2}{x-y},$$

$$\therefore 2x : a :: 2y : \frac{2y^2}{x-y} :: x - y : y.$$

16. $(a-c)x = (d-b)y, \therefore x = \frac{d-b}{a-c} \cdot y \propto y.$

17. *Wood*, art. 212.

$$1455 = (2 \times 5 + 29b)\,15,$$

$$\therefore b = \tfrac{81}{29}.$$

18. Let the series be $2, 2r, 2r^2, 2r^3, 2r^4$,

$\therefore 2r^4 = 32$, and $r^2 = 4, r = \pm 2$, consequently the required means are

$$4,\ 8,\ 16,$$

$$\text{or } -4,\ 8,\ -16.$$

19.

$$\sqrt{(7 - 2\sqrt{10})} = \sqrt{\frac{7+\sqrt{9}}{2}} - \sqrt{\frac{7-\sqrt{9}}{2}} = \sqrt{5} - \sqrt{2}$$

$$\sqrt{(4\sqrt{3} + 7)} = \sqrt{\frac{7+1}{2}} + \sqrt{\frac{7-1}{2}} = 2 + \sqrt{3}.$$

20. Let x = number of seven-shilling pieces, y that of half-guineas; then

$$14x + 21y = 7 \times 40,$$

$$\text{or, } 2x + 3y = 40,$$

$$\therefore x = 20 - y - \frac{y}{2}.$$

Let $y = 0, 2, 4, 6, 8, 10, 12, 14,$

$\therefore x = 20, 17, 14, 11, 8, 5, 2, -1.$

So that the bill may be paid by 20, 0; 17, 2; 14, 4; 11, 6; 8, 8; 2, 12.

21. *Barlow's* Theory of Numbers, p. 261.

22. *Cambridge Mathematical Repository*, p. 21.; and *Barlow's* Theory of Numbers, p. 17.

TRINITY COLLEGE, 1829.

1.

$\frac{1}{19}$ of a £. $= \frac{20}{19}s.$; $\frac{1}{20}$ of a guinea $= \frac{21}{20}s.$, $\frac{8}{35}$ of a crown $= \frac{8}{7}$ of a shilling.

$\therefore$ the values are as $\frac{20}{19}$, $\frac{21}{20}$, $\frac{8}{7}$, or as 2800, 2793, 3040.

2.

$\frac{7}{13}$ of $2s.\ 6d. = \frac{35}{13}$ sixpences, and half-a-guinea $= 21$ sixpences,

$$\therefore \ \frac{35}{13 \times 21} = \text{fraction required} = \frac{5}{39}.$$

Also $10s. = \frac{1}{2}£. = \cdot 5$, $1\frac{1}{2}d. = \frac{1}{8} \times .05 = .00625$,

$$\therefore \ 10s.\ 1\tfrac{1}{2}d. = .50625l.$$

3.

$$2\frac{7}{8} \div \frac{13}{5} = \frac{23}{8} \times \frac{5}{13} = \frac{115}{124}. \text{ Also } 14.4 \div .0012 = \frac{144000}{12} = 12000$$

$$\text{and } \frac{14.\ 4}{120} = \frac{144}{1200} = \frac{12}{100} = .12.$$

4.
$$\frac{3\sqrt{8} - 2\sqrt{7}}{\sqrt{8} - \sqrt{7}} = \frac{(3\sqrt{8} - 2\sqrt{7})(\sqrt{8} + \sqrt{7})}{8 - 7}$$

$$= 24 - 2\sqrt{56} + 3\sqrt{56} - 14 = 10 + \sqrt{56},$$

$$= 17.4833148 \ . \text{ and not } 2 \ . \ 51 \ . \ . \ . \ .$$

The question is exactly the same in the original paper.

Generally let r be the greatest integer in $\sqrt[3]{\text{N}}$, and suppose c the correction ; then

$$\sqrt[3]{\text{N}} = r + c,$$

$$\therefore \text{N} = r^3 + 3r^2 c + 3rc^2 + c^3,$$

$$\therefore \text{N} - r^3 = 3r^2 c + 3rc^2 + c^3,$$

$$\therefore \ r(\text{N} - r^3) + 2rc^3 + c^4 = 3r^3 c + 3r^2 c^2 + 3rc^3 + c^4,$$

$$= 3r^3 c + c(3r^2 c + 3rc^2 + c^3),$$

$$= 3r^3 c + c(\text{N} - r^3) = c(\text{N} + 2r^3),$$

$$\therefore c = \frac{r(\text{N} - r^3)}{\text{N} + 2r^3} + \frac{2rc^3 + c^4}{\text{N} + 2r^3},$$

$$= \frac{r(\text{N} - r^3)}{\text{N} + 2r^3} \text{ nearly} \ . \ \therefore \sqrt[3]{\text{N}} = r + \frac{r(\text{N} - r^3)}{\text{N} + 2r^3} \text{ nearly.}$$

$$\text{By this form } \sqrt[3]{1010} = 10 + c = 10 + \frac{10(1010 - 1000)}{1010 + 2000},$$

$$= 10 + \tfrac{10}{301} = 10.\ 0332228,$$

which is true much further than required by the question; for c is evidently less than 10. Consequently the part omitted, viz.

$$\frac{2rc^3+c^4}{N+2r^3} \text{ is } < \frac{20+\frac{1}{10}}{1000\times 3010} < \frac{201}{30100000}.$$

5. *Wood*, art. 86.

6. By actual division

$$\frac{x^n-y^n}{x-y}=x^n+x^{n-1}y+x^{n-2}y^2+\ldots xy^{n-1}+y^n,$$

as may easily be shown, and the proof is thence derived.

Otherwise.

Suppose it true for $\dfrac{x^{n-1}-y^{n-1}}{x-y}$; that is, let $x^{n-1}-y^{n-1}$ be divisible by $x-y$. Then

$$\begin{array}{l} x-y)\; x^n-y^n\; (x^{n-1} \\ \underline{x^n-x^{n-1}y} \\ \underline{x^{n-1}y-y^n} \end{array}$$

$$\therefore \frac{x^n-y^n}{x-y}=x^{n-1}+y.\frac{x^{n-1}-y^{n-1}}{x-y},$$

$\therefore$ if $x^{n-1}-y^{n-1}$ be divisible by $x-y$, x^n-y^n is divisible by $x-y$. But x^2-y^2 is divisible by $x-y$, $\therefore$ x^3-y^3 is, $\therefore$ x^4-y^4 is, &c. Q. E. D.

Again, suppose, $x^{2n-2}-y^{2n-2}$ divisible by $x+y$; then,

$$\begin{array}{l} x+y)\; x^{2n}-y^{2n}\; (x^{2n-2}-x^{2n-2}y \\ \underline{x^{2n-1}+x^{2n-1}y} \\ \phantom{x+y)\; x^{2n}} -x^{2n-1}y-y^{2n} \\ \phantom{x+y)\; x^{2n}} \underline{-x^{2n-1}y-x^{2n-2}y^2} \\ \phantom{x+y)\; x^{2n}-x^{2n}} y^2(x^{2n-2}-y^{2n-2}) \end{array}$$

$$\therefore \frac{x^{2n}-y^{2n}}{x+y}=x^{2n-1}-x^{2n-2}y+y^2.\frac{x^{2n-2}-y^{2n-2}}{x+y},$$

$\therefore$ if $x^{2n-2}-y^{2n-2}$ is divisible by $x+y$,

$x^{2n}-y^{2n}$ is divisible by $x+y$.

But $x^2 - y^2$ is divisible by $x + y$,
$\therefore\ x^4 - y^4$ is; $\therefore\ x^6 - y^6$, and so on.

Q. E. D.

7. See *Private Tutor,* Alg. Part I. p. 103. et seq.

$3a^5 - 48ab^4)\ 2a^4 - 11a^2b^2 + 12b^2\ ($

$a^4 - 16b^4)\ 2a^4 - 11a^2b^2 + 12b^2$ (rejecting $3a$ from divisor,

$a^2 - 4b^2)\ 2a^4 - 11a^2b^2 + 12b^2$ (rejecting $a^2 + 4b^2$

$\qquad 2a^4 - 8a^2b^2 \qquad\qquad 2a^2 - 3b^2$

$\qquad - 3a^2b^2 + 12b^2$

$\qquad - 3a^2b^2 + 12b^2$, $\quad \therefore$ G. C. M $= a^2 - 4b^2$.

8. *Wood,* art. 106. and 109.

9. (1) $20 - 4x + 8 = 5x + 10$,

$$\therefore\ x = 2.$$

(2) $\left.\begin{array}{r} 7x - y = 33 \\ 12y - x = 19 \end{array}\right\}$,

$\therefore\ 7x - y = 33$

$12y - 7x = 133$,

$\therefore\ 11y = 166$, $\therefore\ y = \frac{166}{11} = 15\frac{1}{11}$,

$\therefore\ 7x = y + 33 = 105\frac{7}{11} + 33 = 138\frac{7}{11}$.

$\therefore\quad x = 19\frac{5}{7} + \frac{1}{11} = 19\frac{62}{77}$.

(3) $x^2 - \frac{3}{2}x = 27$,

$\therefore\ x = \frac{3}{4} \pm \sqrt{(\frac{9}{16} + 27)} = \frac{3}{4} \pm \sqrt{\frac{441}{16}}$

$$= \frac{3 \pm 21}{4} = 6, \text{ or } -\tfrac{9}{2}.$$

(4) $2x^2 - 5x + 6 + 10\sqrt{(2x^2 - 5x + 6)} = 39$,

$\therefore\ \sqrt{(2x^2 - 5x + 6)} + 5 = \pm\sqrt{(39 + 25)} = \pm 8$,

$\therefore\ \sqrt{(2x^2 - 5x + 6)} = 3$ and -13,

$\therefore\quad 2x^2 - 5x + 6 = 9$ and 169,

$\therefore\ x^2 - \frac{5}{2}x = \frac{3}{2}$ and $\frac{163}{2}$,

$\therefore\ x - \frac{5}{4} = \pm\sqrt{(\frac{25}{16} + \frac{3}{2})}$ and $\pm\sqrt{(\frac{25}{16} + \frac{163}{2})}$

$$= \pm\tfrac{7}{4} \text{ and } \frac{\pm\sqrt{1329}}{4},$$

$\therefore x = 3, -\frac{1}{2}, \frac{5+\sqrt{1329}}{4}$ and $\frac{5-\sqrt{1329}}{2}$.

(5) $x^4 - 18x^2 - 4x + 48 = 0$,

$\therefore x^4 - 14x^2 + 49 = 4x^2 + 4x + 1$,

which are perfect squares.

$\therefore x^2 - 7 = 2x + 1$,

$\therefore x^2 - 2x = 8, \therefore x = 1 \pm 3 = 4$ or -2.

Hence, $\frac{x^4 - 18x^2 - 4x + 48}{x^2 - 2x - 8} = 0 = x^2 + 2x - 6$,

$\therefore x + 1 = \pm\sqrt{7}$.

So that the four values of x are $4, -2, 1 \pm \sqrt{7}$.

(6) $x = 22 - y - \frac{6y-6}{11}$.

Let $6y - 6 = 11w$,

$$\therefore y = 1 + w + \frac{5w}{6},$$

which is an integer when

$w = 0, 6, 12, 18, 24, 30 \ldots\ldots$

and then $y = 1, 12, 23, 34, 45, 56 \ldots\ldots$

$x = 21, 4$, the rest being negative,

$\therefore$ the only values of x, y are 21, 1; and 4, 12.

10. Let x be the number of hours required.

cause $\propto \frac{\text{work}}{\text{time}}$. $\propto \frac{1}{\text{time}}$ by the question.

$$\therefore \frac{1}{x} = \frac{1}{x+6} + \frac{1}{x+15} + \frac{1}{2x},$$

$$\therefore \frac{1}{2x} = \frac{1}{x+6} + \frac{1}{x+15} = \frac{2x+21}{(x+6)(x+15)},$$

$$\therefore x^2 + 21x + 90 = 4x^2 + 42x,$$

$$\therefore x + \tfrac{7}{2} = \pm\tfrac{13}{2},$$

$$\therefore x = 3,$$

the only answer, the negative value being -10.

11. In this question the "*p and q together*," should have been "*p and p together*."

Since p and p together gives $\dfrac{n.\,(n-1)\,\ldots\,(n-p+1)}{1.\,2\,\ldots\ldots\,p}$,

and $(n-p)$ and $(n-p)$ together gives

$$\frac{n\,(n-1)\,\ldots\ldots\,n-(n-p-1)}{1.\;\;2.\;\;\ldots\ldots\;\;(n-p)} = \frac{n\,(n-1)\,\ldots\,(p+1)}{1.\;\;2\,\ldots\,(n-p)},$$

$$= \frac{n.\,(n-1)\,\ldots\ldots\,(p+1)\times p\,(p-1)\,\ldots\ldots\,3.\;2.\;1}{1.\;\;2.\;\;\ldots\ldots\;\;(p-1)\,p}$$

$$\times\frac{n\,(n-1)\,\ldots\ldots\,(n-p+2)\;(n-p+1)}{2.\,3\,..\,(n-p-1)\,(n-p)\times(n-p+1)\,(n-p+2)\,..\,(n-1)\,n}$$

$$= \frac{n\,(n-1)\ldots(n-p+2)\,(n-p+1)}{1.\;2\,\ldots\ldots\,(p-1)\,p}\times\frac{1.\;2.\;3\,\ldots\,(n-1)\,n}{1.\;2.\;3\,\ldots\,(n-1)\,n}$$

$$= \frac{n\,(n-1)\,\ldots\ldots\,(n-p+2)\;(n-p+1)}{1.\;2\;\ldots\;(p-1)\,p}.$$

which proves the proposition.

12. See *Private Tutor*, Alg. Part I. p. 21., and 199.

13. See *Private Tutor*, Alg. Part. I. p. 20.

14. $$\frac{1+2x}{1+n} = \frac{1}{\dfrac{1+x}{1+2x}} = \frac{1}{\dfrac{1+2x-x}{1+2x}} = \frac{1}{1-\dfrac{x}{1+2x}},$$

$$\therefore \left(\frac{1+2x}{1+x}\right)^n = \left(1-\frac{x}{1+2x}\right)^{-n} = 1+n\,\frac{x}{1+2x}$$

$$+\frac{n.\,(n+1)}{1\;.\;2}.\;\left(\frac{x}{1+2x}\right)^2 + \text{\&c.}$$

15. $$\sqrt{(7+2\sqrt{10})} = \sqrt{\frac{7+3}{2}} + \sqrt{\frac{7-3}{2}} = \sqrt{5}+\sqrt{2}.$$

Also $\{\sqrt{(4+3\sqrt{-20})} + \sqrt{(4-3\sqrt{-20})}\}^2$

$$= 8+2\sqrt{(16+180)} = 8+28 = 36,$$

$$\therefore\;\sqrt{(4+3\sqrt{-20})} + \sqrt{(4-3\sqrt{-20})} = \pm\,6.$$

16. *Wood*, art. 378.

17. $$N = a_0 + a_1.\,10 + a_2.\,10^2 + a_3.\,10^3 + \ldots\ldots$$

$$0 = a_0 - a_1.\;\;3 + a_2\;3^2 - a_3\,3^3 + \ldots$$

$$\therefore\;N = a_1.\,13 + a_2.\,(10^2-3^2) + a_3\,(10^3+3^3) + a_4\,(10^4-3^4) + \ldots$$

But $10^2 - 3^2$, $10^3 + 3^3$, $10^4 - 3^4$, $10^5 + 3^5$, &c. are all divisible by $10+3$ or 13.

Hence N is divisible by 13.

18. Let $2n$ be the even number. Then
$(2n)^3 - 4.(2n) = 8n^3 - 8n = 8n(n^2-1) = 8.(n-1)n(n+1)$.
But $(n-1)n(n+1)$ is divisible 1. 2. 3, or by 6,
$\therefore (2n)^3 - 4.(2n)$ is by 6×8, or by 48.

19. Since $\left.\begin{array}{r} 2x + 3y = 35 \\ 2\times 4 + 3\times 9 = 35 \end{array}\right\} \therefore 2(x-4) + 3.(y-9) = 0,$
which is of the form,

$$2x' + 3y' = 0,$$
$$\therefore x' = -\frac{3y'}{2}.$$
$$\left.\begin{array}{l} \text{Let } y' = 2w, \text{ or } y = 9 + 2w \\ \therefore x' = -3w, \text{ or } x = 4 - 3w \end{array}\right\},$$

which forms contain all the values of x and y.

For making $w = -4, -3, -2, -1, 0, 1$, we get
$$x = 16, 13, 10, 7, 4, 1$$
$$y = 1, 3, 5, 7, 9, 11.$$

The method applies generally; thus, if
$x = \alpha$, $y = \beta$ satisfy the equation,
$ax + by = c$; then it may easily be shown, that
$$\left.\begin{array}{l} x = \alpha - bw \\ y = \beta + aw \end{array}\right\}.$$

Examples. (1) $18x - 23y = 4$ is satisfied by $x = 13$, $y = 10$.
$$\therefore x = 13 + 23w = 13, 36, 59, 82, 105, 128 \ldots$$
$$y = 10 + 18w = 10, 28, 46, 64, 82, 100 \ldots$$
(2) $25x - 16y = -7$ is satisfied by $x = 1$, $7 = 2$,
$$\therefore x = 1 + 16w = 1, 17, 33, 49, 65, 81 \ldots$$
$$y = 2 + 25w = 2, 27, 52, 77, 102, 127 \ldots$$

20. $\text{Log } 2 = \frac{1}{4} \log 16 = .30103,$
$$\log 30 = \log 2 + \log 15 = \left.\begin{array}{r} .30103 \\ 1.1760913 \end{array}\right\} = 1.471213,$$
$\therefore \log 3 = .471213$. Hence $\log 27 = 3 \log 3 = 1.413639$.
Also $\log 4\frac{1}{20} = \log\frac{81}{20} = 4\log 3 - \log 2 - 1 = 1.884852 - 1.30103$
$$= .583822.$$

21. *Wood,* art. 400, and 402.

22. Let x be the required number of white balls; then the whole number of balls is $x + 11$. The number of ways in which all four can be drawn red $= \frac{11.\ 10.\ 9.\ 8}{1.\ 2.\ 3.\ 4}$, and the whole number of ways in which four balls may be drawn is

$$\frac{(x+11)\,(x+10)\,(x+9)\,(x+8)}{1.\quad 2.\quad 3.\quad 4}.$$

$\therefore$ the probability that all four are red

$$= \frac{11.\quad 10.\quad 9.\quad 8}{(x+11)\,(x+10)\,(x+9)\,(x+8)}.$$

Again, if of m things, m' be of one kind, and these be taken p and p together, the number of these combinations in which p' of the m' things are contained, is (see *Francœur's* Pure Math. art. 478.)

$$\{(m-m')\ \mathrm{C}\ (p-p')\} \times (m'\ \mathrm{C}\ p'),$$

$$\text{or } \frac{(m-m')\,(m-m'-1)\,\ldots\,\{m-m'-(p-p'-1)\}}{1 \quad 2. \quad \ldots \quad (p-p')}$$

$$\times \frac{m'.\,(m'-1)\,\ldots\,(m'-p'-1)}{1.\quad 2\quad \ldots\quad p'}.$$

Hence the number of ways in which 2 of the 11 red balls can be drawn, taking the balls 4 and 4 at a time, is

$$\frac{x.\,(x-1)}{1.\quad 2} \times \frac{11.\quad 10}{1.\quad 2.} = \frac{55\,x\,(x-1)}{2},$$

$\therefore$ the probability that of the four balls, two will be red and two white, is

$$\frac{55\,x\,(x-1)}{2} \times \frac{2.\quad 3.\quad 4}{(x+11)\,(x+10)\,(x+9)\,(x+8)}$$

$$= \frac{11 \times 60\,x(x-1)}{(x+11)\,(x+10)\,(x+9)\,(x+8)}.$$

Hence by the question

$$\frac{11.\quad 10.\quad 9.\quad 8}{(x+11)(x+10)(x+9)(x+8)} = \frac{11 \times 60\,x\,(x-1)}{(x+11)(x+10)(x+9)(x+8)}$$

$$x^2 - x = 3 \times 4 = 12,$$

$$\therefore x = \tfrac{1}{2} \pm \sqrt{(12 + \tfrac{1}{4})} = \frac{1 \pm 7}{2} = 4$$

the number required.

TRINITY COLLEGE, 1830.

1. The sum is $2548\,\frac{41}{60}$.

2. $\frac{1}{21}$ of a pound $= \frac{20}{21}s.$,
$\frac{1}{22}$ of a guinea $= \frac{21}{22}s.$,
$\frac{1}{4}$ of $3s.\ 9\frac{1}{2}d.$ $= \left(3\ \frac{9\frac{1}{2}}{12}\right)\frac{1}{4} = \frac{91}{96}s.$,

$\therefore$ they are as $\frac{20}{21}, \frac{21}{22}, \frac{91}{96}$,
or as $11 \times 48 \times 20,\ 21 \times 21 \times 48,\ 91 \times 11 \times 21$,
or as 10560, 21168, 21021.

3. $\frac{3}{5}$ of a groat $= \frac{1}{5}$ of a shilling $= \frac{2}{5}$ of a sixpence $= \frac{2}{25}$ of half-a-crown.

4. If $= .0625 \times 21s. = 1.3125s. = 1s.\ 3\frac{3}{4}d.$

$$\begin{array}{r} 12 \\ \hline 3.7500 \\ 4 \\ \hline 3.0000. \end{array}$$

5. See *Wright's* Pure Arith. Decimals.

6.
(1) It $= \dfrac{\{\sqrt{(x^2+1)} + \sqrt{(x^2-1)}\}^2 + \{\sqrt{(x^2+1)} - \sqrt{(x^2-1)}\}^2}{2}$

$= 2\,x^2.$

(2) The greatest common measure is $a^2 - b^2$, and the fraction in its lowest terms is

$$\frac{3\,a^2 + 2\,b^2}{5\,a\,(2\,a + 3\,b)}.$$

(3) It $= \left\{\left(x - \frac{1}{x}\right)^2 + 2\left(x - \frac{1}{x}\right)\right\}^{\frac{1}{2}}$

$= \sqrt{\left(x - \frac{1}{x}\right)}\sqrt{\left(x - \frac{1}{x} + 2\right)},$

$= \dfrac{\sqrt{(x^4 + 2x^3 - 2x^2 - 2x + 1)}}{x}.$

(4)

$$\sqrt{(-3-2\sqrt{2})} = \sqrt{\frac{-3+\sqrt{(9-8)}}{2}} - \sqrt{\frac{-3-\sqrt{(9-8)}}{2}}$$
$$= \sqrt{-1} - \sqrt{-2}.$$

(5) It $= \left(\frac{a^{-2m}}{b^{-2n}}\right)^{\frac{pq}{2mn}} = \left(\frac{b^{\frac{1}{m}}}{a^{\frac{1}{n}}}\right)^{pq} = \left(\frac{b^{npq}}{a^{mpq}}\right)^{\frac{1}{mn}}$

the best form for numerical operation.

7. The quotient is $a^6 - 2\,a^{12}\,b^{\frac{1}{3}} + 4\,a^{24}\,b^{\frac{2}{3}} -$ &c.

8. *Wood*, art. 109, 110.

9. (1) $x = 2.$

(2) $\left.\begin{array}{l} x = 2 \\ y = 3 \end{array}\right\}.$

(3) $x = 12$, and $-\ 21$.

(4) $$(x+y)^4 = x^4 + 4\,x^3\,y + 6x^2y^2 + 4\,x\,y^3 + y^4$$
$$= 97 + 4\,xy\,(x^2+y^2) + 6\,x^2\,y^2,$$
$$(x+y)^2 = x^2 + y^2 + 2\,xy,$$
whence $x^2\,y^2 - 50\,xy = -\ 264.$

Hence, the equations become

$$\left.\begin{array}{r} x\,y = 44 \text{ and } 6 \\ \text{and } x+y = 5 \end{array}\right\}.$$

From square of the second, take $4\,x\,y$, which will give

$$\left.\begin{array}{r} x - y = \pm\sqrt{-151}, \text{ and } \pm 1 \\ \text{and } x + y = 5 \end{array}\right\},$$

Hence, $x = \frac{1}{2}\,(5 \pm \sqrt{-151})$, and 3 and 2

$y = \frac{1}{2}\,(5 \mp \sqrt{-151})$, and 2 and 3.

(5) $$x = 30 - y - \frac{3y-3}{5}.$$

Make $\therefore\ y - 1 = 0, 5, 10, 15, 20,$ &c.

$\therefore\ y = 1, 6, 11, 16, 21,$ &c.

and $x = 29, 21, 13, 5, -\ 3,$ &c.

10. Let x be the number of half-pence spent, and y that of the oranges; then,

$$\frac{x}{y} = \text{cost of each},$$

$\therefore$ by the question,

$$\left.\begin{aligned} \frac{x}{y+5} &= \frac{x}{y} - 1 \\ \text{and } \frac{x}{y-3} &= \frac{x}{y} + 1 \end{aligned}\right\} \quad \begin{aligned} x &= 6\tfrac{9}{16} \text{ half-pence } 3\tfrac{1}{4}d.\ \tfrac{1}{16}, \\ y &= 3\tfrac{3}{4}. \end{aligned}$$

11. (1) It is arithmetic, and the common difference $= 6$,

$S = (6 + 6 \times 12)\frac{13}{2} = 13 \times 39 = 487.$

(2) Is geometric, the ratio being $-\frac{3}{2}$.

$$S = \frac{ar^n - a}{r-1} = \frac{\frac{1}{3}(-\frac{3}{2})^5 - \frac{1}{3}}{-\frac{5}{2}} = 1\tfrac{7}{48}.$$

(3) This is arithmetic, the common difference being $-\frac{1}{5}$.

$\therefore S = -(\frac{4}{3} - 5.\frac{1}{5})\,3 = 4 - 3 = 1.$

Also $\therefore .0132132\ldots = \frac{1}{10} \times .132132$, &c.

Make $x = .132132$, &c.

$\therefore 1000\,x = 132 + x,$

$\therefore x = \frac{132}{999} = \frac{44}{333},$

$\therefore .0132132\ldots = \frac{22}{1665}.$

12. Similarly to the question in p. 47.

13. There are eighteen letters, consisting of three a's, b, two i's, three n's, o, r, two s's, four t's, u. But of n things, if k be of one kind, k' of another, k'' of another, &c.; then the number of different permutations is (*Francœur's* P. M. art. 483.)

$$\frac{1.\,2.\,3.\,\ldots\ldots(n-2)\,(n-1)\,n}{1.2.3\ldots(k-1)k \times 1.2.3\ldots(k'-1)k' \times 1.2.3\ldots(k''-1)k'' \times \&c.}$$

Hence, the number of different ways required, is

$$\frac{1.\,2.\,3.\,4.\,5.\,6.\,7.\,8.\,9.\,10.\,11.\,12.\,\ldots\ldots 17.\,18.}{2.\,3 \times 2 \times 2.\,3 \times 2 \times 2.\,3.\,4}$$
$$= 44100 \times 8570 \times 4896$$
$$= 1,850,549,652,000.$$

14. *Private Tutor*, Alg. part i. p. 21, and p. 200.

15. *Private Tutor*, Alg. part i. p. 23.

16. *Barlow's* Theory of Numbers, p. 317.

17. The reciprocals of the terms of an harmonic series being an arithmetic series, let these reciprocals be

$$\tfrac{1}{2},\ \tfrac{1}{2}+x,\ \tfrac{1}{2}+2x,\ \tfrac{1}{2}+3x,\ \tfrac{1}{2}+4x,\ \tfrac{1}{2}+5x,$$

$$\text{then } \tfrac{1}{2}+5x = \tfrac{1}{12};$$

$$\therefore x = -\tfrac{1}{12},$$

$$\therefore \text{ the series is}$$

$$2,\ \frac{2}{1+2x},\ \frac{2}{1+4x},\ \frac{2}{1+6x},\ \frac{2}{1+8x},\ 12,$$

$$\text{or, } 2,\ \tfrac{12}{5},\ 3,\ 4,\ 6,\ 12.$$

18. $n^2 + (n+1)^2 - 5 = 2n^2 + 2n - 4 = 2(n^2 + n - 2)$
$$= 2.(n-1)(n+2).$$

Now, $\because$ n is prime to 3, it must be of the form $3w+1$, or $3w+2$, but $n+1$ is also prime to 3; $\therefore$ n can only be of the form $3w+1$.

Hence,

$n^2+(n+1)^2-5$ is of the form $2.3w.3(w+1$, or of $18.w(w+1)$, which is divisible by 36, for either w or $w+1$ must be even.

19. Let $n = 2w$;

$$\text{then, } \frac{n^3+20n}{48} = \frac{8w^3+40w}{48} = \frac{w^3+5w}{6} = \frac{w.(w^2+5)}{6}$$

$$\text{But } w = 3w',\ 3w'+1, \text{ or } 3w'+2,$$

$$\therefore \frac{n^3+20n}{48} = \frac{3w'(9w'^2+5)}{6} \text{ or } \frac{(3w'+1)(9w'^2+6w'+6)}{6}.$$

$$\text{or } \frac{(3w'+2)(9w'^2+12w'+9)}{6}$$

$$= \frac{w'(9w'^2+5)}{2} \text{ or } \frac{(3w'+1)(3w'^2+2w'+2)}{2}$$

$$\text{or } \frac{(3w'+2)(3w'^2+4w'+3)}{2},$$

each of which is divisible by 2, when w' is either even or odd.

20. *Wood*, art. 378.

21. $\text{Log}\,5 = \log 10 - \log 2 = 1 - \frac{1}{2}\log 4 = 1 - .30103 = .69897.$

22. Let x be the time the first college has it, and A be the rent charge. By *Wood*, art. 402.

$$P = \frac{1 - \frac{1}{R^x}}{R - 1} A, \text{ and it } = \frac{A}{R-1} \text{ when } x \text{ is } \infty,$$

$$\therefore \frac{A}{R-1} - \frac{1 - \frac{1}{R^x}}{R-1} A = \frac{1 - \frac{1}{R^x}}{R-1} A, \text{by the question.}$$

Hence, $2.\left(1 - \frac{1}{R^x}\right) = 1$, $\therefore R^x = 2$, and $x = \frac{\log 2}{\log R} = \frac{\log 2}{\log \frac{21}{20}}$

$$= \frac{\log 2}{\log 105 - \log 100}$$

$$= \frac{\log 2}{.0211893}$$

$$= \frac{.3010300}{.0211893}$$

$$= 14\,.\,25 \text{ nearly,}$$

$$= 14\,.\,\tfrac{1}{4} \text{ years, nearly.}$$

23. Assume $a^x = A + Bx + Cx^2 + Dx^3 + \text{\&c.}$

A, B, C, &c. being independent of x,

$$a^{2x} = A + B(2x) + C(2x)^2 + D(2x)^3 + \text{\&c.}$$

$$= A^2 + 2ABx + B^2 x^2 + 2BCx^3 + \text{\&c.}$$

$$2ACx^2 + 2ADx^3 + \text{\&c.}$$

$$\therefore A^2 = A,\ 2AB = 2B,\ B^2 + 2AC = 4C,$$

$$8D = 2BC + 2AD,$$

$$\therefore A = 1,\ B = B,\ C = \frac{B^2}{1.2},\ D = \frac{B^3}{1.2.3},\ \text{\&c.}$$

$$\therefore a^x = 1 + Bx + \frac{B^2}{1.2}x^2 + \frac{B^3}{1.2.3}x^3\ \text{\&c.}$$

$$\text{Let } x = \frac{1}{B},$$

$$\therefore a^{\frac{1}{B}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots = e, \text{suppose.}$$

$$\therefore B = \frac{\log a}{\log e},$$

which gives the series.

wrong. See Errata.

24. The probabilities due to the assertions of A, B, and C, are respectively

$$\tfrac{3}{4}, \tfrac{4}{5}, \tfrac{6}{7},$$

$\therefore$ the probability required is $\frac{3}{4} + \frac{4}{5} - \frac{6}{7} = \frac{97}{140}$.

ALGEBRA.

PART II.

TRINITY COLLEGE, 1819.

[P. 50.]

1. *Wood,* art. 266.

2. *Wood,* art. 271. Two typographical errors in this question. In first line of it, for *terms* read *term,* and for *least term* read *last* term.

3. Let the roots be of the form $\frac{a}{r}$, a, ar; then

$$\frac{a}{r} \times a \times ar = 8, \therefore a = 2,$$

which being one of the roots, the other two may thence be found by the solution of a quadratic. But this may be done directly; for

$$\frac{2}{r} + 2 + 2r = 7,$$

$$\therefore 2r^2 - 5r = -2,$$

$$\therefore r^2 - \tfrac{5}{2}r = -1,$$

$$\therefore r - \tfrac{5}{4} = \pm \tfrac{3}{4},$$

$$\therefore r = 2, \text{ or } \tfrac{1}{2},$$

$$\therefore \text{the roots are } 1, 2, 4.$$

4. *Wood,* art. 277.

5. "Or equal," it ought to have been added. See *Wood*, art. 331., and *Private Tutor*, Alg. part ii. p. 71.

6. Since the reciprocals of quantities in harmonic progression are in arithmetic progression, if $y = \frac{1}{x}$, the roots of the equation in y will be in arithmetic progression. √ That is, the roots of

$$\frac{1}{y^3} - \frac{22}{y^2} + \frac{144}{y} - 288 = 0,$$

$$\text{or } y^3 - \tfrac{1}{2}y^2 + \frac{11}{144}y - \frac{1}{288} = 0.$$

Of this equation let $u - v$, u, $u + v$, be the roots,

$$\therefore\ u = \tfrac{1}{6},$$

$$\therefore\ v = \pm \tfrac{1}{12},$$

$\therefore$ the roots of the equation in y are

$$\tfrac{1}{6} - \tfrac{1}{12},\ \tfrac{1}{6},\ \tfrac{1}{6} + \tfrac{1}{12},$$

$$\text{or } \tfrac{1}{12},\ \tfrac{1}{6},\ \tfrac{1}{4},$$

$\therefore$ the roots of the given equation on

12, 6, 4.

7. (1) One root in each is respectively -3 and 1.

(2) First deprive it of the second term, and then use the rule in *Wood*.

8. See *Private Tutor*, Alg. part ii. p. 202. The new equation is

$$y^3 - \frac{p^2+q}{pq-r}y^2 + \frac{2p}{pq-r}y - \frac{1}{pq-r} = 0.$$

N. B.—The second term should be $-px^2$, and not $-qpx^2$.

9. The equation of limits has one of them; that is

$$3x^2 + \tfrac{20}{7}x = 0,$$

$$\therefore \text{ the equal roots are } -\tfrac{20}{21},$$

$$\therefore \text{ the other root is } \tfrac{10}{21}.$$

10. The equation is Reciprocal; $\therefore\ b = \dfrac{1}{a}$.

Also $a + b = a + \dfrac{1}{a} = p$

$$p^n = \left(a + \frac{1}{a}\right)^n = a^n + n\,a^{n-2} + \frac{n\,(n-1)}{1.\ 2}\,a^{n-4} + \ldots$$

$$\frac{n\,(n-1)}{1.\ 2}\,\frac{1}{a^{n-4}} + n.\frac{1}{a^{n-2}} + \frac{1}{a^n},$$

$$= a^n + \frac{1}{a^n} + n\left(a^{n-2} + \frac{1}{a^{n-2}}\right) + \frac{n.\,(n-1)}{1.\ 2}\left(a^{n-4} + \frac{1}{a^{n-4}}\right) + \ldots$$

$$= a^n + b^n + n\,(a^{n-2} + b^{n-2}) + \frac{n\,(n-1)}{1.\ 2}\,(a^{n-4} + b^{n-4}) + \ldots$$

making $S_n = a^n + b^n$, $S_{n-2} = a^{n-2} + b^{n-2} + \ldots$

$$S_n = p^n - n\,S_{n-2} - \frac{n\,(n-1)}{1.\ 2}\,S_{n-4} - \frac{n\,(n-1)(n-2)}{1.\ 2.\ 3}\,S_{n-6}, - \text{\&c.}$$

Similarly $S_{n-2} = p^{n-2} - (n-2)\,S_{n-4} - \dfrac{(n-2)\,(n-3)}{1.\ 2}\,S_{n-6} - \text{\&c.}$

$$\therefore S_n = p^n - np^{n-2} + \left\{n\,(n-2) - \frac{n\,(n-1)}{1.\ 2}\right\}\,S_{n-4}$$

$$+ \left\{\frac{n\,(n-2)\,(n-3)}{1.\ 2} - \frac{n\,(n-1)\,(n-2)}{1.\ 2.\ 3}\right\}\,S_{n-6} + \text{\&c.}$$

But $n\,(n-2) - \dfrac{n\,(n-1)}{1.\ 2} = \dfrac{n}{2}\,(2n-4-n+1) = \dfrac{n\,(n-3)}{1.\ 2}$

and $\dfrac{n\,(n-2)\,(n-3)}{1.\ 2} - \dfrac{n\,(n-1)\,(n-2)}{1.\ 2.\ 3} = \dfrac{n\,(n-2)}{1.\ 2.\ 3}\,(3n-9-n+1),$

$$= \frac{n\,(n-2)}{1.\ 2.\ 3}\,(2n-8)$$

$$= \frac{n\,(n-2)\,(n-4)}{1.\ 3}$$

$$\therefore S_n = p^n - np^{n-2} + \frac{n\,(n-3)}{1.\ 2.}\,S_{n-4} + \frac{n\,(n-2)(n-4)}{1.\ 3}\,S_{n-6} + \text{\&c.}$$

Similarly,

$$S_{n-4} = p^{n-4} - (n-4)\,p^{n-6} + \frac{(n-4)\,(n-7)}{1.\ 2}\,S_{n-8}$$

$$+ \frac{(n-4)\,(n-6)\,(n-8)}{1.\ 3}\,S_{n-10} + \text{\&c.}$$

$$S_{n-6} = p^{n-6} - (n-6)\,p^{n-8} + \frac{(n-6)\,(n-9)}{1.\ 2}\,S_{n-10}$$

$$+ \frac{(n-6)(n-8)(n-10)}{1.\ \ 3} S_{n-12} + \&c.$$

$$\therefore S_n = p^n - np^{n-2} + \frac{n(n-3)}{1.\ 2} p^{n-4} -$$

$$\left\{\frac{n(n-3)}{1.\ 2.}(n-4) - \frac{n(n-2)(n-4)}{1.\ \ 3}\right\} p^{n-6} + \&c.$$

$$= p^n - np^{n-2} + \frac{n(n-3)}{1.\ 2} p^{n-4} - \frac{n(n-4)(n-5)}{1.\ 2.\ 3} p^{n-6} + \&c.$$

Otherwise.

Assume the formula true for $a^{n-1} + b^{n-1}$, and also for $a^{n-2}+b^{n-2}$, and then multiplying $a^{n-1}+b^{n-1}$ by $a+b$, prove it true for a^n+b^n. Then, it being true for a^2+b^2 and a^3+b^3, it is for a^4+b^4, $\therefore$ for a^5+b^5, and so on.

11. For the proof of *Waring's* rule, see *Wood*, art. 344.

The reducing cubic being

$$8n^3 + 4qn^2 + (8s - 4rp)n + 4qs + 4p^2s - r^2 = 0,$$

and $\because p = -3,\ q = -12,\ r = 12,\ s = -4,$

$$\therefore 8n^3 - 48n^2 + (-32 + 144)n + 192 - 144 - 144 = 0,$$

$$\therefore \quad n^3 - 6n^2 + 14n - 12 = 0,$$

of which a root is evidently 2. Substituting this in

$$(x^2 + px + n)^2 = (p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s,$$

we have

$$(x^2 - 3x + 2)^2 = x^2 + 0 + 4 - 4 = x^2,$$

$$\therefore x^2 - 3x + 2 = \pm x,$$

$$\left.\begin{array}{r} \therefore x^2 - 4x = -2 \\ \text{and } x^2 - 2x = -2 \end{array}\right\},$$

$$\left.\begin{array}{r} \therefore x = 2 \pm \sqrt{2} \\ x = 1 \pm \sqrt{-1} \end{array}\right\} \text{the four roots.}$$

12. *Wood*, art. 311.

13. A superior limit is 262. *Wood*, art. 304.

Otherwise.

Since $x^4 - 19x^3 + 117x^2 - 261x + 162 = 0$,
Make $x = y + e$, and substitute; then

$\therefore\ 0 = y^4 + (4e - 19)y^3 + (6e^2 - 57e + 117)y^2 + (4e^3 - 57e^2$
$+ 234e - 261)y -$
$+ e^4 - 19e^3 + 117e^2 - 261e + 162,$

in which every coefficient being positive when $e = 10$, 10 is $>$ the greatest root of the equation.

Again, to find the Inferior Limit, since all the roots are positive, (*Wood*, art. 311.), if we make $x = \frac{1}{u}$, the Superior Limit of the equation in u will give us the Inferior Limit of the equation in x. That is, the Superior Limit of

$$u^4 - \tfrac{29}{18}u^3 + \tfrac{13}{18}u^2 - \tfrac{19}{162}u + \tfrac{1}{162} = 0$$

will give an Inferior Limit of the given equation.

But u is $< \frac{29}{18} + 1 < \frac{47}{18}$,

$$\therefore \tfrac{1}{u} \text{ is } > \tfrac{18}{47}$$

$\therefore\ x$ is $> \frac{18}{47}$; that is, $\frac{18}{47}$ is less than the least root.

The roots actually are 9, 6, 3, 1.

14. Dividing the equation by a^2x^2, we have

$$\left(\frac{x^2}{a^2} + \frac{a^2}{x^2}\right) + m\left(\frac{x}{a} + \frac{a}{x}\right) + n = 0,$$

$$\text{or, } \left(\frac{x}{a} + \frac{a}{x}\right)^2 + m\left(\frac{x}{a} + \frac{a}{x}\right) = -n + 2,$$

$$\therefore \frac{x}{a} + \frac{a}{x} = \frac{-m \pm \sqrt{(m^2 - 4n + 8)}}{2} = c, \text{ suppose,}$$

whence the root is easy.

15. The new equation being

$$y^n + \frac{p}{2}y^{n-1} + \frac{q}{4}y^{n-2} - \frac{r}{8}y^{n-3} + \&c. = 0.$$

But let $u = x \times -\frac{1}{2}$, or $x = -2u$,

$$\therefore\ \mp 2^n u^n \mp p\,2^{n-1}u^{n-1} \mp q\,2^{n-2}u^{n-2} \pm r.\,2^{n-3}u^{n-3} \mp \&c. = 0.$$

$$\text{or, } u^n + \frac{p}{2}u^{n-1} + \frac{q}{4}u^{n-2} - \frac{r}{8}u^{n-3} + \&c. = 0.$$

$\therefore$ the values of y = the values of u = those of $-\dfrac{x}{2}$

$$= -\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}, \frac{d}{2}, \frac{e}{2}, \&c.$$

TRINITY COLLEGE, 1821.

1. *Wood*, art. 277.

2. Since $\sqrt{-5}$ is one root of the equation, $-\sqrt{-5}$ must be another, and the equation is

$$(x-3)(x-2)(x-\sqrt{-5})(x+\sqrt{-5}) = 0,$$
$$\text{or } (x^2-5x+6)(x^2+5) = 0,$$
$$\text{or } x^4-5x^3+11x^2-25x+30 = 0.$$

Again, since $\sqrt{2}$ is a root of the next equation, $-\sqrt{2}$ must also be a root, and the equation is divisible by $(x-3)(x-\sqrt{2}) \times (x+\sqrt{2})$, or by $(x-3)(x^2-2)$, or by x^3-3x^2-2x+6, and the quotient is

$$x^2-7x+10 = 0,$$
$$\therefore \text{ the roots are } 2,\ 3,\ 5,\ \sqrt{3},\ -\sqrt{3}.$$

3. Make $y = \dfrac{1}{x}$, and the equation in y, whose roots are in Arithmetic Progression, is

$$y^3-y^2+26y-24 = 0.$$

Let the roots of this be $u-v$, u, $u+v$; then

$$u-v+u+u+v = 9 = 3u \therefore u = 3,$$
$$\text{and } (u^2-v^2)\,u = 24 \therefore 9-v^2 = 8, \text{ and } v = 1,$$
$$\therefore \text{ the roots required are}$$
$$\tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4}.$$

4. See *Wood*, art. 311.

5. The coefficients of the equation in y, if $y = x + e$ are

$$\begin{array}{r}
e^5 + 2e^4 - 50e^3 - 100e^2 + 49e + 98, \\
5e^4 + 8e^3 - 150e^2 - 200e + 49, \\
10e^3 + 12e^2 - 150e - 100, \\
+ 10e^2 + 8e - 50, \\
+ 5e + 2,
\end{array}$$

all of which being positive when $e=8$, 8 is a *Superior Limit* of the roots. Also, it is found that 7 satisfies the equation. $\therefore$ 7 is the greatest root.

Again, it is evident there are three negative roots, there being that number of continuations of sign. See *Wood*, art. 311. Hence, changing the signs of the roots, the Superior Limit of the new equation

$$x^5 - 2x^4 - 50x^3 + 100x^2 + 49x - 98 = 0$$

will be the Inferior Limit required.

Here the coefficients of y of the equation in $y = x + e$, are

$$\begin{array}{r} e^5 - 2e^4 - 50e^3 + 100e^2 + 49e - 98, \\ 5e^4 - 8e^3 - 150e^2 + 200e + 49, \\ 10e^3 - 12e^2 - 150e + 100, \\ 10e^2 - 8e - 50, \\ 5e - 2. \end{array}$$

These are all positive when $e = 8$, and the first is 0, when $e = 7$, $\therefore -8$ is a limit less than the least root of the given equation, and -7 is that least root.

6. (1) First deprive the equation of its second term, by making $y = x - \frac{6}{3} = x - 2$, and it becomes

$$y^3 - 9y - 28 = 0.$$

But by *Cardan's* rule,

$$y = \sqrt[3]{\left[-\frac{r}{2} + \sqrt{\left\{\left(\frac{r}{2}\right)^2 - \left(\frac{q}{3}\right)^3\right\}}\right]}$$
$$+ \sqrt[3]{\left[-\frac{r}{2} - \sqrt{\left\{\left(\frac{r}{2}\right)^2 - \left(\frac{q}{3}\right)^3\right\}}\right]},$$

and $\frac{q}{3} = 3$ and $\frac{r}{2} = -14$,

$$\therefore y = \sqrt[3]{\{-14 + \sqrt{(196-27)}\}} + \sqrt[3]{\{14 - \sqrt{(196-27)}\}}$$
$$= \sqrt[3]{27} + \sqrt[3]{1} = 3 + 1 = 4,$$

$\therefore x = 6$ one of the roots.

Whence the other two are $\sqrt{3}$ and $-\sqrt{3}$.

(2) Let a, a, b, b, be the equal roots.

$$\therefore 2a + 2b = -p,$$
$$a^2 b^2 = s,$$
$$\therefore a + b = -\frac{p}{2},$$

$$ab = \pm\sqrt{s}.$$

$$\text{Hence } \therefore a = -\frac{p}{4} \pm \frac{1}{4}\sqrt{(p^2 \mp 16\sqrt{s})},$$

$$b = -\frac{p}{4} \mp \frac{1}{4}\sqrt{(p^2 \mp 16\sqrt{s})}.$$

Otherwise.

The given equation and limiting equation have a common measure of the form $x^2 + Px + Q = 0$, which contains the equal roots

$$4x^3 + 3px^2 + 2qx + r)\, 4x^4 + 4px^3 + 4qx^2 + 4rx + 4s\,(x + p$$
$$\underline{4x^4 + 3px^3 + 2qx^2 + rx}$$
$$px^3 + 2qx^2 + 3rx + 4s,$$
$$\text{or } 4px^3 + 8qx^2 + 12rx + 16s$$
$$\underline{4px^3 + 3p^2x^2 + 2pqx + pr}$$
$$\therefore (8q - 3p^2)x^2 + (12r - 2pq)x + 16s - pr,$$

consequently the roots are those of the equation,

$$x^2 + \frac{12r - 2pq}{8q - 3p^2}x = \frac{pr - 16s}{8q - 3p^2},$$

which are easily found.

The first method is preferable, because of its involving only two of the coefficients p and s. Take the numerical example

$$x^4 + 2x^3 - 3x^2 - 4x + 4 = 0.$$

By the first method,

$$a = -\tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{(4 \mp 32)} = -\tfrac{1}{2} \pm \tfrac{6}{4} = 1 \text{ or } -2,$$
$$b = -\tfrac{1}{2} \mp \tfrac{3}{2} = -2 \text{ or } 1,$$

$\therefore$ the roots are $1, 1, -2, -2$.

By the other method, the equation containing the roots, is

$$x^2 + \frac{-48 + 12}{-24 - 12}x = \frac{-8 - 64}{-24 - 12},$$

$$\text{or } x^2 + x = 2$$

$$\therefore x = 1 \text{ or } -2, \text{ as before.}$$

(3) Divide by a^2x^2, and the equation becomes

$$\frac{x^2}{a^2} + \frac{x}{a} + 1 + \frac{a}{x} + \frac{a^2}{x^2} = 0.$$

Make $\frac{x}{2} + \frac{a}{x} = u$; then the equation becomes

$$u^2 + u - 1 = 0,$$

$$\therefore u = -\frac{1}{2} \pm \frac{\sqrt{5}}{2},$$

$$\text{or } \frac{x}{a} + \frac{a}{x} = \frac{-1 \pm \sqrt{5}}{2},$$

$$\therefore \left(\frac{x}{a}\right)^2 - \frac{-1 \pm \sqrt{5}}{2} \cdot \frac{x}{a} = -1,$$

$$\text{and } \frac{x}{a} = \frac{-1 \pm \sqrt{5}}{4} \pm \sqrt{\frac{(-1 \pm \sqrt{5})^2 - 16}{16}},$$

$$\therefore x = \frac{a}{4} \{-1 \pm \sqrt{5} \pm \sqrt{(\mp 2\sqrt{5} - 10)}\},$$

and the four roots are

$$x = \frac{a}{4} \{-1 + \sqrt{5} + \sqrt{(-2\sqrt{5} - 10)}\},$$

$$\text{or} = \frac{a}{4} \{-1 - \sqrt{5} + \sqrt{(2\sqrt{5} - 10)}\},$$

$$\text{or} = \frac{a}{4} \{-1 + \sqrt{5} - \sqrt{(-2\sqrt{5} - 10)}\},$$

$$\text{or} = \frac{a}{4} \{-1 - \sqrt{5} - \sqrt{(2\sqrt{5} - 10)}\},$$

all of which are imaginary, and may be reduced to the form $A + B\sqrt{-1}$.

7. Since ~~S~~ a is a root of the equation,

$$S + Ra + Qa^2 + Pa^3 + \ldots \, p\, a^{n-1} + a^n = 0,$$

$$\therefore \frac{S}{a} + R + Qa + Pa^2 + \ldots \, p\, a^{n-2} + a^{n-1} = 0,$$

$$\therefore \frac{S}{a} = -R - Qa - Pa^2 - \ldots - p\, a^{n-2} - a^{n-1},$$

$\therefore$ S is divisible by a,

$$\text{also, } \because R_1 = \frac{S}{a} + R = -Qa - Pa^2 - \ldots - p\, a^{n-2} - a^{n-1},$$

$$\therefore \frac{R_1}{a} = -Q - Pa - \ldots - p\, a^{n-3} - a^{n-2},$$

that is, R_1 is divisible by a,

$$\therefore Q_1 = \frac{R_1}{a} + Q = -Pa - \ldots - p\, a^{n-4} - a^{n-3},$$

$\therefore$ Q_1 is divisible by a,

&c.

whence the meaning of the proposition is manifest. It is imperfectly enumerated, however; for it is by no means necessary, although S, R_1, Q_1, &c. may be divisible by a, that the quotients are *integers*.

8. This symmetrical function of the roots =

$$a^2(b^3+c^3+d^3+e^3+f^3+g^3)+b^2(a^3+c^3+\ldots .)+\&c.$$
$$=a^2(S_3-a^3)+b^2(S_3-b^3)+c^3(S_3-c^3)+\&c.$$
$$=S_3\times S_2-S_5,$$

S_2 meaning the sum of the squares of the roots,

S_3 that of the cubes, &c.

But, generally, if the equation be (see *Wood*, art. 352)

$$x^n+p\,x^{n-1}+q\,x^{n-2}+\&c.=0,$$
$$S_1+p=0,\ S_2+p\,S_1+2\,q=0,\ S_3+p\,S_2+q\,S_1+3\,r=0,$$
$$S_4+p\,S_3+q\,S_2+r\,S_1+4\,s=0,$$
$$S_5+p\,S_4+q\,S_3+r\,S_2+s\,S_1+5\,t=0,$$

from which five *simple* equations S_1, S_2, S_3, S_4, S_5, may be found, and the value of $S_3\times S_2-S_5$ may be found.

But, eliminating S_1, S_2, S_3, &c. we have

$$S_1=-p\ \ldots\ldots\ \quad (1)$$
$$\therefore\ S_2=p^2-2q\ \ldots\ldots\ \quad (2)$$
$$\therefore\ S_3=-p^3+3pq-3r\ \ldots\ldots\ \quad (3)$$
$$\therefore\ S_4=p^4-4p^2q+4pr+2q^2-4s\ \ldots\ \quad (4)$$
$$\therefore\ S_5=-p^5+5p^3q-5p^2r-5pq^2+5ps-5qr-5t\ldots (5)$$

&c.

These are particular cases of *Waring's* general Theorem. See Meditationes Alg. Theorem 1.

In this case,

$$p=-6,\ q=9,\ r=10,\ s=-45,\ t=54,$$
$$\therefore\ S_2=18,$$
$$S_3=-408,$$
$$S_5=-684,$$
$$\therefore\ S_2=18,\ \ S_3=-408,\ S_5=-684,$$
$$\therefore\ S_2\times S_3-S_5=-8028,\text{ the answer.}$$

9. Taking the equation of limits, as in *Wood*, art 352,

$$\frac{n\,x^{n-1}-(n-1)\,px^{n-2}+(n-2)\,yx^{n-3}-\&c.}{x^n-px^{n-1}+qx^{n-2}-\&c.}$$

$$=\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\&c.$$

$$=\frac{1}{x}+\frac{a}{x^2}+\frac{a^2}{x^3}+\frac{a^3}{x^4}+\&c.$$

$$\frac{1}{x}+\frac{b}{x^2}+\frac{b^2}{x^3}+\frac{b^3}{x^4}+\&c.$$

n series,

$$=\frac{n}{x}+S_1.\frac{1}{x^2}+S_2.\frac{1}{x^3}+S_3.\frac{1}{x^4}+\&c.\ldots\infty$$

Let $x=1$; then,

$$S_1+S_2+S_3+\ldots.\infty=\frac{n-(n-1)p+(n-2)\,q-\ldots}{1-p+q-\&c.}-n$$

$$=\frac{p-2q+3r-4s+5t-\&c.}{1-p+q-r+s-t+\&c.}.$$

This is the answer usually given to the present question. But the student must observe, that unless every one of the roots of the equation be a proper fraction, and, consequently, S_1, S_2, S_3, &c. a *converging* series, the result cannot obtain *arithmetically*. For example, in the equation $x^2-px+1=0$, we shall have $S_1+S_2+\ldots.\infty=\dfrac{p-2}{1-p+1}=-1$, which is absurd; for the roots being of the form α, $\dfrac{1}{\alpha}$, if they are real, oné of them must be an *improper* fraction, and, consequently, the sum of its successive powers to ∞ must be *infinite*, not considering the sum of the powers of its reciprocal.

In short, the question itself is absurd, except for the equations whose roots are *all* less than unity.

10. Assume the required equation to be

$$u^3+\mathrm{P}\,u^2+\mathrm{Q}\,u+\mathrm{R}=0.$$

Then $-\mathrm{P}=a\,\alpha+a\beta+b\,\beta=a\,(\alpha+\beta)+b\,\beta=-a\,p_1+b\,\beta$,

$$\mathrm{Q}=a^2\,\alpha\,\beta+\,a\,b\,\alpha\,\beta+ab\,\beta^2=a^2\,q_1+q\,q_1+q\,\beta^2,$$

$$-\mathrm{R}=a^2\,b\,\alpha\,\beta^2=a\,\beta.\,q\,q_1,$$

and solving the given equations, a, b will be found in terms of p, q; and α, β will be obtained in terms of p_1, q_1; which being substituted in the values of P, Q, R, these last will be found, and, consequently, the required equation.

It will be observed, the roots of the required equation are not symmetrical. The question would have been better, if the roots of the required equation had been of the form

$$a\alpha,\ a\beta,\ b\beta,\ b\alpha.$$

For then, in the assumed equation,

$$u^4+\mathrm{P}u^3+\mathrm{Q}u^2+\mathrm{R}u+\mathrm{S}=0,$$

$$\begin{aligned}
-\mathrm{P} &= a.(\alpha+\beta)+b(\alpha+\beta)=(a+b)(\alpha+\beta)=pp_1,\\
\mathrm{Q} &= a^2\alpha\beta+ab\,\alpha\beta+ab\,\alpha^2+ab\,\beta^2+ab\,\alpha\beta+b^2\alpha\beta,\\
&= (a^2+b^2)\alpha\beta+(\alpha^2+\beta^2)ab+2qq_1=(p^2-2q)q_1\\
&\qquad + (p_1^2-2q)q+2qq_1,\\
&= p^2q_1+p_1^2q-2qq,\\
-\mathrm{R} &= a\alpha.\,a\beta.\,b\beta+a\alpha.\,a\beta.\,b\alpha+a\alpha.\,b\beta.\,b\alpha+a\beta.\,b\beta.\,b\alpha\\
&= a^2b\,\alpha\beta^2+a^2b\,\alpha^2\beta+ab^2\,\alpha^2\beta+ab^2\,\alpha\beta^2,\\
&= qq_1\,a\beta+qq_1\,a\alpha+qq_1\,b\alpha+qq_1\,b\beta,\\
&= qq_1a(\alpha+\beta)+qq_1b(\alpha+\beta)=qq_1(a+b)(\alpha+\beta)=pp_1qq_1\\
\mathrm{S} &= a\alpha.\,a\beta.\,b\beta.\,b\alpha=a^2b^2.\,\alpha^2\beta^2=q^2q_1^2,
\end{aligned}$$

$\therefore$ the equation is, in this case of symmetrical roots,

$$u^4-pp_1u^3+(p^2q_1+p_1^2q-2qq_1)u^2-pp_1qq_1u+q^2q_1^2=0.$$

11. See *Wood*, 327.

12. See *Wood*, art. 332.

13. (1) *Wood*, art. 278.

14. See *Private Tutor*, Alg. part. ii.

15. See *Private Tutor*, Alg. part. ii.

16. If the roots be real, there are as many changes of sign from positive to negative, as positive roots, and as many continuations as negative roots; hence, all those consecutive terms which are wanting, being + or −, and indicating

changes or continuations indifferently, show the imaginary roots. Hence, if the equation be

$$x^n - px^{n-1} + \ldots \pm Px^p \ldots \pm Qx^{p-2m-1} + \&c. = 0,$$

in which $2m$ terms are wanting, then between

$$\pm Px^p \pm 0.\, x^{p-1} \pm 0.\, x^{p-2} \pm \ldots \pm 0.\, x^{p-2m} \pm Qx^{p-2m-1},$$

there are indifferently $2m$ changes, or $2m$ continuations, which indicate $2m$ imaginary roots. $\therefore$ if $2m$ terms are wanting, the equation has $2m$ imaginary roots. Again, between

$$\pm P\, x^p.\ \pm 0.\, x^{p-1} \pm 0.\, x^{p-2} \pm \ldots \pm 0.\, x^{p-2m-1} \pm Rx^{p-2m-2},$$

in which $2m+1$ terms are wanting. If P and R have the same sign, these are indifferently $2m+2$ changes and continuations, and $\therefore$ $2m+2$ imaginary roots. But if P and R have different signs, then there are indifferently but $2m$ changes or continuations, and $\therefore$ but $2m$ imaginary roots.

Examples.

1. $x^2 - q = 0$, has two real roots.
2. $x^3 - r = 0$, has one real root.
3. $x^4 - s = 0$, has but two imaginary roots.
4. $x^4 + s = 0$, has $2m+2$, or four imaginary roots.
5. $x^5 - t = 0$, has four imaginary roots.
6. $x^6 + w = 0$, has six imaginary roots.
7. $x^{10} + x^7 + x^5 + x + 1$, has two imaginary roots indicated by $x^{10} + x^7$; and four indicated by $x^5 + x$.

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1. See *Private Tutor*, Alg. part ii. p. 45.

2. *Wood*, art. 277, and 275.

3. The requadratic is

$$(x-\sqrt{3})\,(x+\sqrt{3})\,(x+\sqrt{-5})\,(x-\sqrt{-5}) = 0,$$
$$\text{or } (x^2-3)\,(x^2+5) = 0,$$
$$\text{or } x^4+2\,x^2-15 = 0.$$

Again, since one root of the given equation is of the form $\sqrt{a}$, another must be of the form $-\sqrt{a}$.

Let $\therefore$ the three roots be

$$\sqrt{a},\ -\sqrt{a} \text{ and } b. \quad \text{Then}$$

$$\sqrt{a}.\ -\sqrt{a}+b=4, \text{ or } b=4,$$

and $\sqrt{a}.\ -\sqrt{a}\times -b=12$, $\therefore$ $a=3$, and the roots are

$$\sqrt{3},\ -\sqrt{3} \text{ and } 4.$$

4. The sum is

$$\begin{array}{rcrcrcr}
a & + & 2b & + & 3c & + & 4d \\
a & + & 2b & + & 4c & + & 3d \\
a & + & 3b & + & 2c & + & 4d \\
a & + & 3b & + & 4c & + & 2d \\
a & + & 4b & + & 2c & + & 3d \\
a & + & 4b & + & 3c & + & 2d \\
2a & + & b & + & 3c & + & 4d \\
2a & + & b & + & 4c & + & 3d \\
2a & + & 3b & + & c & + & 4d \\
2a & + & 3b & + & 4c & + & d \\
2a & + & 4b & + & c & + & 3d \\
2a & + & 4b & + & 3c & + & d \\
3a & + & b & + & 2c & + & 4d \\
3a & + & b & + & 4c & + & 2d \\
3a & + & 2b & + & c & + & 4d \\
3a & + & 2b & + & 4c & + & d \\
3a & + & 4b & + & c & + & 2d \\
3a & + & 4b & + & 2c & + & d \\
4a & + & b & + & 2c & + & 3d \\
4a & + & b & + & 3c & + & 2d \\
4a & + & 2b & + & c & + & 3d \\
4a & + & 2b & + & 3c & + & d \\
4a & + & 3b & + & c & + & 2d \\
4a & + & 3b & + & 2c & + & d \\
\hline
60\,(a & + & b & + & c & + & d)
\end{array}$$

The number of these forms will evidently be that of the permutations in four things taken all together. Consequently, 4. 3. 2. 1 = 24 is the number of such forms. Also, a, $2a$, $3a$, $4a$ will each occur $3\times 2\times 1$, or six times; therefore the

number of a's in the required sum is $6\ (a+2a+3a+4a) = 60\,a$, and the sum itself is evidently $\therefore$

$$60\ (a+b+c+d).$$

But by the Method of Divisors, it is easily seen, that the roots are $1, -1, 2, 3$. Consequently $d=1$, and the sum required is

$$60\ (1+2+3+1) = 420.$$

Again, if a, b, c, d, be the roots of the equation,

$$x^4 - p\,x^3 + q\,x^2 - r\,x + s = 0,$$

$$ab + ac + ad + bc + bd + cd = q,$$

$$\text{and } abcd = s,$$

$$\therefore \frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} = \frac{q}{s},$$

$$\therefore \frac{1}{a^2b^2} + \frac{1}{a^2c^2} + \frac{1}{a^2d^2} + \frac{1}{b^2c^2} + \frac{1}{b^2d^2} + \frac{1}{c^2d^2} = \frac{q^2}{s^2}$$

$$- 2\left\{\frac{1}{ab\,ac} + \frac{1}{ab\,ad} + \frac{1}{ab\,bc} + \frac{1}{ab\,bd} + \frac{1}{ab\,cd} + \frac{1}{ac\,cd}\right.$$

$$+ \frac{1}{ac\,bc} + \frac{1}{ac\,bd} + \frac{1}{ac\,cd} + \frac{1}{ad\,bc} + \frac{1}{ad\,bd} + \frac{1}{ad\,cd}$$

$$\left. + \frac{1}{bc\,bd} + \frac{1}{bc\,cd} + \frac{1}{bd\,cd}\right\}$$

$$= \frac{q^2}{s^2} - 2\left\{\frac{1}{a}\left(\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}\right)\right.$$

$$+ \frac{1}{b}\left(\frac{1}{abc} + \frac{1}{abd} + \frac{1}{bcd}\right)$$

$$+ \frac{1}{c}\left(\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}\right)$$

$$\left. + \frac{1}{d}\left(\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}\right)\right\}$$

$$= \frac{q^2}{s^2} - 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)\left(\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}\right)$$

$$+ \frac{2}{ab\,cd}$$

$$= \frac{q^2}{s^2} - 2.\frac{r}{s}.\frac{p}{s} + \frac{2}{s} \quad (\textit{Wood}, \text{ art. } 273.)$$

$$= \frac{q^2 - 2pr + 2s}{s^2}.$$

In the present example,

$$p = 5,\ q = 5,\ r = -5, \text{ and } s = -6,$$

$\therefore$ the sum required is

$$\frac{25 + 50 - 12}{144} = \frac{63}{144} = \frac{7}{16}.$$

5. Since the reciprocals of roots in Harmonic Progression are in Arithmetic Progression, let

$$\frac{1}{a-b},\ \frac{1}{a},\ \frac{1}{a+b},$$

be the roots of the given equation; then, (*Wood*, art. 273.)

$$a-b+a+a+b = \frac{q}{r};\ \therefore a = \frac{q}{3r}, \text{ and } \therefore \frac{1}{a} = \frac{3r}{q},$$

consequently, $x^3 - p\,x^2 + q\,x - r = 0$ is divisible by

$x - \dfrac{3r}{q}$, the quotient being

$$x^2 - \frac{p\,q - 3\,r}{q}\,x + \frac{q^3 - 3\,p\,q\,r + 9\,r^2}{q^2} = 0,$$

which, consequently, contains the roots $\dfrac{1}{a-b}$, and $\dfrac{1}{a+b}$, viz. the greatest and least. Hence, those roots are easily found; or they may be obtained independently of the quadratic.

$$\text{For, } \frac{1}{a^2 - b^2}.\ \frac{1}{a} = r,$$

$$\text{whence, } b = \frac{\pm\sqrt{(q^3 - 27\,r^2)}}{3\,r\sqrt{q}},$$

&c.

6. (1) Let a, b, c, be the roots required; then,

$$\left.\begin{array}{r} a+b = 13 \\ a+b+c = 15 \\ a\,b\,c = 80 \end{array}\right\} \therefore c = 2,\quad \left.\begin{array}{r} a\,b = 40 \\ a+b = 13 \end{array}\right\} \text{ whence,}$$

$$\therefore a = 8, \text{ or } 5,$$
$$b = 5, \text{ or } 8.$$

(2) Let $2\,a$, $3b$, c, be the roots; then,

$$\left.\begin{array}{r} 5\,a + c = 17 \\ 2\,a \times 3a + 2\,ac + 3\,ac = 94 \end{array}\right\},$$

$$\left.\begin{array}{r}\text{or,}\quad 5a+c=17\\ 6a^2+5ac=94\end{array}\right\},\qquad \therefore\ 19a^2-85a=-94$$
$$a^2-\tfrac{85}{19}a=-\tfrac{94}{19},$$
$$\therefore\ a=2,\ \text{or}\ \tfrac{47}{19},$$
$$c=7,\ \text{or}\ \tfrac{86}{19},\ \text{whence the roots are found.}$$

(3) The equation and its equation of limits having a common measure of the form $(x-a)^2\times(x-b)$, if this be found, a and b may also be found very easily.

Otherwise.

$$3a+2b=13,$$
$$a^2+a^2+ab+ab+a^2+ab+ab+ab+ab+b^2=67,$$
$$\left.\begin{array}{r}\text{or,}\ 3a+2b=13\\ 3a^2+6ab+b^2=67\end{array}\right\},\qquad \text{whence, } a=3,\ b=2.$$

(4) The general form of triangular numbers being $\frac{n.(n+1)}{1.\,2}$, let the three roots be $\frac{n\,(n+1)}{2}$, $\frac{(n+1)(n+2)}{2}$, $\frac{(n+2)\,(n+3)}{2}$. Then, $\frac{3n^2+9n+8}{2}=31$,

which gives $n=3$, or -6,

and the roots are 6, 10, 15,

or 15, 10, 6,

according as n is made $=3$, or -6.

7. (1) Since 1 is a root (*Wood*, art. 326.) dividing the equation by $x-1$, and the quotient, viz.

$$x^8-(p-1)x^7+(q-p+1)x^6-(r-q+p-1)x^5$$
$$+(s-r+q-p+1)x^4-(r-q+p-1)x^3+(q-p+1)x^2$$
$$-(p-1)x+1=0,$$

contains the other roots. Divide by x^4; then,

$$x^4+\frac{1}{x^4}-(p-1)\left(x^3+\frac{1}{x^3}\right)+(q-p+1)\left(x^2+\frac{1}{x^2}\right)$$
$$-(r-q+p-1)\left(x+\frac{1}{x}\right)$$
$$+s-r+q-p+1=0.$$

Make $x+\dfrac{1}{x}=u$,

$$\therefore\ x^2+\frac{1}{x^2}=u^2-2,$$

$$x^3+\frac{1}{x^3}=u^3-3u,$$

$$x^4+\frac{1}{x^4}=(u^2-2)^2-2=u^4-4u^2+2,$$

$$\therefore\ u^4-(p-1)u^3+(q-p-3)u^2+(2p+q-r-3)u$$
$$+2(p-q)=0.$$

(2) The roots are $1,\ \dfrac{-1+\sqrt{-3}}{2},\ \dfrac{-1-\sqrt{-3}}{2},$

or $1,\ \dfrac{-1+\sqrt{-3}}{2},\ \dfrac{2}{-1+\sqrt{-3}}.$

(See *Private Tutor*, Alg. part ii. p. 399.)

8. The root found by the rule is 4. The rule fails when two of the roots are neither impossible nor equal. (*Wood*, art. 331. *Private Tutor*, Alg. vol. ii. part ii. p. 71.)

9. Since $(A+B)^3 = A^3+B^3+3AB(A+B)$

$$= -r+3\left\{\frac{r^2}{4}-\left(\frac{r^2}{4}-\frac{q^3}{27}\right)\right\}^{\frac{1}{3}}.(A+B)$$

$$= -r+q(A+B),$$

$$\therefore\ (A+B)^3-q(A+B)+r=0.$$

But the given equation is

$$x^3-qx+r=0,$$

$\therefore$ $A+B$ is a root of the equation.

Again,

$$(\alpha A+\beta B)^3=\alpha^3A^3+\beta^3B^3+3\alpha\beta AB(\alpha A+\beta B)$$

$$= A^3+B^3+3AB(\alpha A+\beta B),$$

for $\alpha^3-1=0,\ \beta^3-1=0$, and $\beta=\dfrac{1}{\alpha}$.

$$\therefore\ (\alpha A+\beta B)^3=-r+q(\alpha A+\beta B),$$

$\therefore$ &c. as is evident. And similarly for the other root.

10. *Private Tutor*, Alg. vol. ii. part ii. p. 121.

11. (1) Let the required equation be

$$y^2 + \mathrm{P}\, y + \mathrm{Q} = 0.$$

Then, $\mathrm{P} = \sqrt{a} + \sqrt{b}$, $\mathrm{Q} = \sqrt{(ab)} = \sqrt{q}$,

$$\therefore \mathrm{P}^2 = a + b + 2\sqrt{(ab)} = p + 2\sqrt{q},$$

$$\therefore y^2 + (p + 2\sqrt{q})\, y + \sqrt{q} = 0$$

is the equation required.

(2) $x^n + q\, x^{n-2} + s\, x^{n-4} + \ldots = a^{\frac{1}{2}} (p\, x^{n-1} + r\, x^{n-3} + \&c.)$.

Let $x = y.\, a^{-\frac{1}{2}}$,

$$\therefore y^n a^{-\frac{n}{2}} + q\, y^{n-2}\, a^{-\frac{n}{2}+1} + s\, y^{n-4}\, a^{-\frac{n}{2}+2} + \&c.$$

$$= a^{\frac{1}{2}} (p\, y^{n-1}\, a^{-\frac{n}{2}+\frac{1}{2}} + r\, y^{n-3}\, a^{-\frac{n}{2}+\frac{3}{2}} + \&c.)$$

$$= p\, y^{n-1}\, a^{-\frac{n}{2}+1} + r\, y^{n-3}\, a^{-\frac{n}{2}+2} + \&c.$$

$$\therefore y^n + q\, a\, y^{n-2} + s\, a^2 y^{n-4} + \&c. = p\, a\, y^{n-1} + r\, a^2 y^{n-3} + \&c.$$

$$\therefore y^n - p\, a\, y^{n-1} + q\, a\, y^{n-2} - r\, a^2 y^{n-3} + s\, a^2 y^{n-4} - \&c. = 0,$$

in which the coefficients are both rational and entire.

(3) Let the required equation be

$$y^3 - \mathrm{P}\, y^2 + \mathrm{Q}\, y - \mathrm{R} = 0,$$

then $\because$ its are roots are (see p. 46),

$$\frac{a+b}{2},\ \sqrt{(ab)} \text{ and } \frac{2ab}{a+b},$$

$$\text{or } \frac{p}{2},\ \sqrt{q} \text{ and } \frac{2q}{p},$$

$$\therefore \mathrm{P} = \frac{p}{2} + \frac{2q}{p} + \sqrt{q},$$

$$\mathrm{Q} = \frac{p}{2}.\sqrt{q} + \frac{p}{2}.\frac{2q}{p} + \frac{2q\sqrt{q}}{p} = \frac{p\sqrt{q}}{2} + q + \frac{2q\sqrt{q}}{p},$$

$$\mathrm{R} = q\sqrt{q},$$

$\therefore$ the required equation is

$$y^3 - \left(\frac{p}{2} + \frac{2q}{p} + \sqrt{q}\right) y^2 + \left(\frac{p\sqrt{q}}{2} + q + \frac{2q\sqrt{q}}{p}\right) y - q\sqrt{q} = 0.$$

12. Since $\left.\begin{aligned} a+b+c &= p \\ a+b+c' &= p' \end{aligned}\right\} \therefore c - c' = p - p'.$

Also,

$$\left.\begin{aligned} a\,b\,c &= r \\ a\,b\,c' &= r' \end{aligned}\right\} \therefore \frac{c}{c'} = \frac{r}{r'},$$

$$\text{whence } c = r.\frac{p-p'}{r-r'}, \text{ and } c' = r'.\ \frac{p-p'}{r-r'}.$$

13. (1) If $y = \dfrac{1}{x}$; then the Superior Positive Limit of y will be the Inferior Positive Limit of x. But the equation in y is

$$y^4 + \tfrac{2}{3}.\, y^3 - \tfrac{5}{3}.\, y^2 + y + \tfrac{1}{3} = 0,$$

and if $u = y + e$, the coefficients of the equation in u are

$$\therefore\ e^4 + \tfrac{2}{3}.\, e^3 - \tfrac{5}{3}.\, e^2 + e + \tfrac{1}{3},$$
$$4\, e^3 + 2\, e^2 - \tfrac{10}{3}.\, e + 1,$$
$$6\, e^2 + 2\, e - \tfrac{5}{3},$$
$$4\, e + \tfrac{2}{3},$$

which are all positive when $e = \frac{1}{2}$, and not so when $e = \frac{1}{3}$, $\therefore\ \frac{1}{2}$ is a *near* Superior Limit of y, or a near Inferior Limit of x.

(2) As in example p. 75, it will be found that 1 is an Inferior Limit of the roots or values of x. Consequently, if all the roots be increased by 1, they will become positive; that is, the terms will be alternately positive and negative. (*Wood*, art. 265.) Making, in fact, $y = x + 1$, we get, by substitution,

$$y^3 - 4\, y^2 + \tfrac{13}{2}\, y - \tfrac{1}{2} = 0.$$

14. The given equation being of the form

$$(x-a)\ (x-b)\ (x-c) \ldots\ldots = 0,$$

that of limits, viz.

$$n\, x^{n-1} - (n-1)\ x^{n-2} + \ldots\ldots = 0 \ldots\ldots (a)$$

is of the form (see *Wood*, art. 308.)

$$(x-a)\ (x-b) \ldots\ldots + (x-a)\ (x-c) \ldots\ldots + \ldots\ldots = 0,$$

each term having $(n-1)$ factors.

Similarly, the limiting equation of (a), viz.

$$n\ (n-1)\ x^{n-2} - (n-1)\ (n-2)\ x^{n-3} + \ldots\ldots = 0 \ldots\ldots (\beta)$$

is of the form

$$(x-a)\ (x-b) \ldots\ldots + (x-a)\ (x-c) \ldots\ldots + \ldots\ldots = 0,$$

each term consisting of $(n-2)$ factors.

Lastly, the limiting equation of (β), viz.

$n(n-1)(n-2)x^{n-3}-(n-1)(n-2)(n-3)x^{n-4}+\&c.=0\ldots(\gamma)$

is of the form

$$(x-a)\,(x-b)\ldots.+(x-a)\,(x-c)\ldots.=0,$$

each term containing $n-3$ factors. Consequently, since the given equation has n factors,

$$\frac{n\,(n-1)\,(n-2)\,x^{n-3}-\&c.}{x^n-p\,x^{n-1}+\&c.}=$$

$$\frac{1}{(x-a)\,(x-b)\,(x-c)}+\frac{1}{(x-a)\,(x-b)\,(x-d)}+\&c.$$

TRINITY COLLEGE, 1823.

1. First find an Inferior Limit of the roots of the given equation, or a Superior Limit of those of the equation

$$y^4-y^3-19\,y^2-11\,y+30=0,$$

whose roots are those of the given equation with their signs changed.

Secondly, augment the roots of the proposed equation by this Limit, and the roots will become all positive, and $\therefore$ the signs of the terms alternately positive and negative.

But the coefficients of the equation in y, when we make

$$y=x+e,$$

$$\begin{aligned}\text{are }\ & e^4-e^3-19\,e^2-11\,e+30,\\ & 4\,e^3-\ 3\,e^2-38\,e-11,\\ & 6\,e^2-\ 3\,e-19,\\ & 4\,e-1,\end{aligned}$$

which are all positive when $e=5$, and not, when $e=4$, $\therefore$ 5 is a *near* Superior Limit of y, or a near Inferior Limit of x. Make $\therefore$ in the original equation

$$u=x+5,\text{ or }x=u-5,$$

and we get

$$\therefore\ u^4-19\,u^3+116\,u^2-224\,u+1=0$$

for the required equation.

N

2. If a be the common root; then $x-a$ is a common measure of the two equations. By the usual rule, this is

$$x - 3.$$

$\therefore$ 3 is the common root; whence the other roots are found by reducing the equations to quadratics;

and the roots are

$$2, 3, 4; -1, 3, 5.$$

3. Let a be the given root of the equation,

$$x^n - px^{n-1} + qx^{n-2} - \&c. = 0,$$

$$\text{then } a^n - pa^{n-1} + qa^{n-2} - \&c. = 0,$$

$$\therefore x^n - a^n - p\,(x^{n-1} - a^{n-1}) + q\,(x^{n-2} - a^{n-2}) - \&c. = 0.$$

But each of these terms being divisible by $x-a$, (see *Private Tutor*, Alg. part i. p. 98.) the equation is reducible to the form

$$x^{n-1} - p_1 x^{n-2} + q_1 x^{n-3} - \ldots = 0,$$

which being supposed to have one root, a_1 may be, in like manner, reduced to an equation of the form

$$x^{n-2} - p_2\, x^{n-3} + q_2\, x^{n-3} - \&c. = 0,$$

and so on.

Whence, it is evident, that if *any* equation have one root, it will have as many as it has dimensions.

This is the meaning of the proposer of the question; for otherwise, it being easy to give any equation a root, by multiplying it by $x-1$, for instance, the general proposition which so many have endeavoured to establish in a simple manner, would be extremely easy.

4. *Wood*, art. 271. Or it may be proved better for Examinations, by assuming it true for m dimensions; then multiplying the equation $x-\lambda$, λ being the $(m+1)$th root, and proving it true for $m+1$ dimensions. But it is true for two dimensions; thence it will be true for three dimensions, and so on.

5. Make $y = \frac{1}{x}$; then the equation in y has the roots $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$; which equation is

$$\therefore\ y^3 - \frac{q}{r} \cdot y^2 + \frac{p}{r} y - \frac{1}{r} = 0.$$

Assume the required equation to be

$$u^3 - Pu^2 + Qu - R = 0,$$

then $P = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{q^2}{r^2} - 2\frac{p}{r}$, *Wood*, art. 273.

$$Q = \frac{1}{a^2 b^2} + \frac{1}{a^2 c^2} + \frac{1}{b^2 c^2} = \frac{a^2 + b^2 + c^2}{a^2 b^2 c^2},$$

$$= \frac{p^2 - 2q}{r^2},$$

$$\text{and } R = \frac{1}{a^2} \cdot \frac{1}{b^2} \cdot \frac{1}{c^2} = \frac{1}{r^2},$$

$\therefore$ the equation required is

$$u^3 - \frac{q^2 - 2pr}{r^2} u^2 + \frac{p^2 - 2q}{r^2} u - \frac{1}{r^2} = 0.$$

6. (1) There being two changes of signs, there are two positive roots, and $\therefore$ three negative roots.

(2) This is departing from the usual definition of a root. But see p. 80.

7. Let $x^m = y$; then

$$y^3 - py^2 + qy - r = 0.$$

Take away the second term of this, by making

$$y = u + \frac{p}{3},$$

and then one root of the equation can be found by *Cardan's* Method, when two of its roots are either *imaginary* or *equal*. (See *Private Tutor*, Alg. part ii. p. 71). Hence may be found, in those cases, the roots of the equation in u; then those of the equation in y; and finally, the equation may be fully resolved by finding each of the three sets of roots of m

roots each contained in the form $x^m - y = 0$. See *Private Tutor,* Alg. part ii. p. 399.

8. (1) *Wood,* art. 304.

(2) The terms being *given* alternately positive and negative, all the roots are supposed positive (*Wood,* art. 265). Hence, making $y = \frac{1}{x}$, the Superior Limit to the roots of the equation in y is an Inferior Limit to those of the equation in x. But the equation in y is

$$y^n \ldots \mp \frac{N}{u} y^r \ldots \pm \frac{P}{u} y^m \ldots \pm \frac{q}{u} y^2 \mp \frac{p}{u} \pm 1 = 0.$$

From this equation a Superior Limit to its roots, is

$$\frac{N}{u} + 1, \text{ or } \frac{P}{u} + 1,$$

according as u is + or − (*Wood,* art. 304); that is,

$$\frac{1}{\frac{N}{u} + 1} \text{ or } \frac{1}{\frac{P}{u} + 1}$$

is an Inferior Limit of the given equation, according as u is positive or negative. Whence the proposition is manifest.

9. *Wood,* art. 320.

10. (1) First -1 is a root. Let the other roots be $a, \frac{1}{a}, b, \frac{1}{b}$; then

$$\left.\begin{aligned} &a + \frac{1}{a} + b + \frac{1}{b} - 1 = 21, \\ &\text{and } a.\frac{1}{a} + ab + \frac{a}{b} - a + \frac{b}{a} + \frac{1}{ab} - \frac{1}{a} + b.\frac{1}{b} - b - \frac{1}{b} = 37 \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{or } a + \frac{1}{a} + b + \frac{1}{b} &= 22 \\ ab + \frac{1}{ab} + \frac{a}{b} + \frac{b}{a} &= 57 \end{aligned}\right\}$$

Let $a + \frac{1}{a} = u, b + \frac{1}{b} = v,$

$$\left.\begin{aligned} u + v &= 22 \\ uv &= 57 \end{aligned}\right\}$$

$$u^2 + 2uv + v^2 = 484$$
$$4uv \quad = 228,$$
$$\therefore u^2 - 2uv + v^2 = 256,$$
$$\therefore \left.\begin{aligned} u - v &= \pm 16 \\ u + v &= 22 \end{aligned}\right\} \therefore \begin{aligned} u &= 19 \text{ or } 3 \\ v &= 3 \text{ or } 19 \end{aligned}$$

$$\text{whence } a = \frac{3 \pm \sqrt{5}}{2}, \frac{1}{a} = \frac{3 \mp \sqrt{5}}{2}$$

$$b = \frac{19 \pm \sqrt{257}}{2},$$

$$\text{and } \frac{1}{b} = \frac{19 \mp \sqrt{257}}{2}.$$

Otherwise.

Dividing the equation by $x + 1$, we get
$$x^4 - 22x^3 + 59x^2 - 22x + 1 = 0.$$
Dividing this by x^2, and arranging
$$x^2 + \frac{1}{x^2} - 22\left(x + \frac{1}{x}\right) + 59 = 0.$$
$$\text{Make } x + \frac{1}{x} = u,$$
$$\therefore u^2 - 22u = -57.$$
Hence u, and $\therefore$ the values of x; as is obvious.

(2) *Private Tutor*, Alg. part ii. p. 25.

As to the question holding good when m is any prime number, that is shown not to be the case in the simplest instance. For one root of $x^3 + 1 = 0$, is -1, of which the other two are $\frac{1 \pm \sqrt{-3}}{2}$.

11. *Private Tutor*, Alg. vol. ii. part ii. p. 79.

12. (1) By *Wood*, art. 338, if for x we put $1, 0, -1$, it is found that -1 satisfies the equation, and is $\therefore$ a root without further trouble. Whence the other roots are found by the reduced quadratic to be 3 and 4.

(2) *Barlow's* Theory of Numbers, p. 32.

13. *Wood*, art. 510.

14. *Wood*, art. 352.

15. Let S_1, S_2, S_3, &c. S_m, S_{m+1}. . . . denote the sums of the simple powers, squares, cubes, &c. of the roots a, b, c, &c. of an equation of n dimensions; then, $\because$

$$S_m = a^m + b^m + \dots.$$
$$S_{m+1} = a^{m+1} + b^{m+1} + \dots.$$

if a be the greatest root,

$$\frac{S_{m+1}}{S_m} = \frac{a^{m+1} + b^{m+1} + \dots.}{a^m + b^m + \dots.}$$

Suppose $b = a.b'$, $c = a.c'$, &c.; then,

$$\frac{S_{m+1}}{S_m} = \frac{a^{m+1}}{a^m} \cdot \frac{1 + b'^{\,m+1} + c'^{\,m+1} + \dots.}{1 + b'^{\,m} + c'^{\,m} + \dots.}$$

$$= a\,\frac{1 + b'^{\,m+1} + c'^{\,m+1} + \dots.}{1 + b'^{\,m} + c'^{\,m} + \dots.}$$

But $\because$ a is the greatest root, $\therefore$ b', c', &c. are each < 1, $\therefore$ the greater m is the more nearly $\dfrac{1 + b'^{\,m+1} + \dots.}{1 + b'^{\,m} + \dots.}$ is equal to unity, and the more nearly to a does $\dfrac{S_{m+1}}{S_m}$ approximate.

Example. *Required to approximate to the greatest root of*

$$x^3 - 3x^2 + 2x - 1 = 0,$$

$$S_1 - p = 0,$$
$$S_2 - pS_1 + 2q = 0,$$
$$S_3 - pS_2 + qS_1 - 3r = 0,$$
$$S_4 - pS_3 + qS_2 - rS_1 = 0,$$
$$S_5 - pS_4 + qS_3 - rS_2 = 0,$$
&c.

In the present case, $p = 3$, $q = 2$, $r = 1$,

$\therefore$ $S_1 = 3$, $S_2 = 5$, $S_3 = 12$, $S_4 = 29$, $S_5 = 68$, $S_6 = 158$, $S_7 = 367$, $S_8 = 853$, $S_9 = 1983$, $S_{10} = 4610$, $S_{11} = 10717$, $S_{12} = 24914$, $S_{13} = 57918$.

Hence, if a be the greatest root of the equation, we have

$$a = \frac{S_2}{S_1} = \frac{5}{3}, \text{ or } = \frac{S_3}{S_2} = \frac{12}{5}, \text{ or } = \frac{29}{12}, \text{ or } \frac{68}{29}, \text{ or } \frac{158}{68},$$

$$\text{or } \frac{367}{158}, \text{ or } \frac{853}{367}, \text{ or } \frac{1983}{853}, \text{ or } \frac{4610}{1983}, \text{ or } \frac{10717}{4610}, \text{ or } \frac{24914}{10717},$$

$$\text{or } \frac{57918}{24914}.$$

To approximate to the least root of an equation in x, make $y = \frac{1}{x}$, and then, as above, approximate to the greatest root of the transformed equation in y.

16. (1) First, make $y = \dfrac{1}{x}$; then, the equation in y is

$$y^5 + \frac{43}{60}y^4 - \frac{19}{40}y^3 - \frac{13}{120}y^2 + \frac{3}{40}y - \frac{1}{120} = 0,$$

whose roots are $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}, \dfrac{1}{d}, \dfrac{1}{e}$.

But, as in the example of p. 94,

$$S_1 - p = 0,$$
$$S_2 - pS_1 + 2q = 0,$$
$$S_3 - pS_2 + qS_1 - r = 0,$$
$$\therefore S_1 = \frac{43}{60}, \quad S_2 = \frac{43^2}{60^2} - \frac{19}{20},$$
$$S_3 = \frac{43}{60}\left(\frac{43^2}{60^2} + \frac{19}{60}\right) + \frac{19}{40}.\frac{43}{60} + \frac{13}{40}.$$

whence the numerical value of S_3 may be found.

(2) The Function $= a^2(b^3+c^3+d^3+e^3)$

$$+ b^2(a^3+c^3+d^3+e^3)$$
$$+ c^3(a^3+b^3+d^3+e^3)$$
$$+ d^3(a^3+b^3+c^3+e^3)$$
$$+ e^3(a^3+b^3+c^3+d^3)$$
$$= a^2(S_3-a^3)+b^2(S_3-b^3)+c^2(S_3-c^3)+ \&c.$$
$$= S_3.S_2 - S_5.$$

But, S_2, S_3, and S_5, may be easily found from the equations

$$S_1 - p = 0,$$
$$S_2 - pS_1 + 2q = 0,$$

$$S_3 - p\,S_2 + q\,S_1 - 3\,r = 0,$$
$$S_4 - p\,S_3 + q\,S_2 - r\,S_1 + 4\,s = 0,$$
$$S_5 - p\,S_4 + q\,S_3 - r\,S_2 + s\,S_1 - 5\,t = 0,$$

p, q, r, s, t, being the coefficients in the general equation,

$$x^5 - p\,x^4 + q\,x^3 - r\,x^2 + s\,x - t = 0.$$

TRINITY COLLEGE, 1824.

1. *Private Tutor*, Alg. part ii. p. 15.

2. Reducing the roots to a common denominator, they are

$$\frac{a^2+b^2}{a^2\,b^2},\quad \frac{a^2+c^2}{a^2\,c^2},\quad \frac{b^2+c^2}{b^2\,c^2}.$$

Assume $\therefore\ y = \dfrac{\dfrac{p^2-2\,q-x^2}{r^2}}{x^2} = x^2.\ \dfrac{p^2-2\,q-x^2}{r^2}$

$$= \frac{p^2-2\,q}{r^2}.\ x^2 - \frac{1}{r^2}.\ x^4,$$

$$\left.\begin{array}{r} \therefore\ x^4 - (p^2-2q)\,x^2 + r^2\,y = 0, \\ \text{and } x^3 + p\,x^2 + q\,x + r = 0, \end{array}\right\}$$

from which x being eliminated, the equation in y will be that required. See *Private Tutor*, Alg. part ii. p. 139.

Otherwise.

Assume the required equation to be

$$y^3 - P\,y^2 + Q\,y - R = 0,$$

then, $P = \dfrac{a^2+b^2}{a^2\,b^2} + \dfrac{a^2+c^2}{a^2\,c^2} + \dfrac{b^2+c^2}{b^2\,c^2}$

$$= \frac{\dfrac{p^2-2\,q-c^2}{r^2}}{c^2} + \frac{\dfrac{p^2-2\,q-b^2}{r^2}}{b^2} + \frac{\dfrac{p^2-2\,q-a^2}{r^2}}{a^2}$$

$$= \frac{(p^2-2q)\,(a^2+b^2+c^2)}{r^2} - \frac{1}{r^2}\,(a^4+b^4+c^4).$$

But the sums of the powers are always known in terms of the coefficients. *Wood*, art. 352. Consequently, P is known, and similarly, Q and R may be found.

3. Let $a = \lambda \cos\theta$, $b = \lambda \sin\theta$, which is allowable, because λ may be of any magnitude, whilst θ may be as small as required,

$$\text{then, } (a+b\sqrt{-1})^{\frac{1}{4}} = \lambda^{\frac{1}{4}}(\cos\theta + \sqrt{-1}.\sin\theta)^{\frac{1}{4}}$$

$$= \lambda^{\frac{1}{4}}\left(\cos\frac{\theta}{4} + \sqrt{-1}\sin\frac{\theta}{4}\right),$$

$$\text{and } (a-b\sqrt{-1})^{\frac{1}{4}} = \lambda^{\frac{1}{4}}\left(\cos\frac{\theta}{4} - \sqrt{-1}.\sin\frac{\theta}{4}\right),$$

$$\left.\begin{aligned} \therefore \sqrt[4]{(a+b\sqrt{-1})} + \sqrt[4]{(a-b\sqrt{-1})} &= 2\lambda\cos\frac{\theta}{4} \\ \sqrt[4]{(a+b\sqrt{-1})} - \sqrt[4]{(a-b\sqrt{-1})} &= 2\lambda.\sqrt{-1}.\sin\frac{\theta}{4} \end{aligned}\right\}$$

in which $\lambda = \sqrt{(a^2+b^2)}$, and $\theta = \cos^{-1}\dfrac{a}{\sqrt{(a^2+b^2)}}$.

4. If all the roots are real, and be called a, b, c, d, &c.; then the factors are $x-a$, $x-b$, $x-c$, &c. and the equation is decomposable into $(x-a)(x-b)$, $(x-c)(x-d)$, &c. or into $x^2-(a+b)x+ab$, $x^2-(c+d)x+cd$, &c. which are real.

If any of them be imaginary, they must consist of pairs of the form $\alpha+\beta\sqrt{-1}$, $\alpha-\beta\sqrt{-1}$, and the factors hence obtained are $(x-\alpha-\beta\sqrt{-1})(x-\alpha+\beta\sqrt{-1})$, or $(x-\alpha)^2+\beta^2$, or $x^2-2\alpha x+\alpha^2+\beta^2$.

5. Since α is a root, $\alpha^n - 1 = 0$, $\therefore \alpha^n = 1$, $\therefore (\alpha^n)^{\frac{p}{n}} = 1$,

$$\therefore \alpha^p - 1 = 0, \text{ or } \alpha^p \text{ is a root.}$$

Let $p = n+m$; then,

$$\alpha^{n+m} - 1 = 0,$$

$$\therefore \alpha^n.\alpha^m - 1 = 0.$$

But $\alpha^n = 1$, $\therefore \alpha^m - 1$, or the roots recur when p is $> n$.

6. (1) This may be done by the Method of Divisors, or by *Cardan's* Rule

1	14	1, 2, 7, 14	2
0	− 9	∓1, ∓3, ∓9	3
− 1	− 4	∓1, ∓2, ∓4	4

o

$\therefore$ 3 is a root. Whence the other two are $\dfrac{\pm\sqrt{3}.\sqrt{-1}-3}{2}$.

By *Cardan's* Method.

$$x = \sqrt[3]{\left[-\frac{r}{2}+\sqrt{\left\{\left(\frac{r}{4}\right)^2-\left(\frac{q}{3}\right)^3\right\}}\right]}$$

$$+\sqrt{\left[-\frac{r}{2}-\sqrt{\left\{\left(\frac{r}{2}\right)^2-\left(\frac{q}{3}\right)^3\right\}}\right]},$$

$$=\sqrt[3]{\left(\frac{9}{2}+\sqrt{\frac{49}{4}}\right)}+\sqrt[3]{\left(\frac{9}{2}-\sqrt{\frac{49}{4}}\right)}$$

$$=\sqrt[3]{8}+\sqrt[3]{1}=3.$$

(2) Let the roots be $\dfrac{a}{r^3}$, $\dfrac{a}{r}$, ar, ar^3. Then

$\dfrac{a}{r^3}\cdot\dfrac{a}{r}\cdot ar.\ ar^3 = s$, or $a^4 = s$, $\therefore\ a = s^{\frac{1}{4}}$.

Again, let $y = \dfrac{x}{a}$, then the equation becomes

$$a^4y^4+pa^3y^3+qa^2y^2+ray+s=0,$$

$$\text{or } y^4+\frac{py^3}{a}+\frac{q}{a^2}y^2+\frac{r}{a^3}y+\frac{s}{a^4}=0,$$

$$\text{or } y^4+\frac{p}{s^{\frac{1}{4}}}y^3+\frac{q}{s^{\frac{1}{4}}}y^2+\frac{r}{s^{\frac{3}{4}}}y+1=0,$$

which being a recurring equation, whose roots are

$$\frac{1}{r^3},\ \frac{1}{r},\ r \text{ and } r^3$$

$$\frac{p}{s^{\frac{1}{4}}}=\frac{r}{s^{\frac{3}{4}}}.\ \ (\textit{Wood}, \text{ art. } 325.)$$

$\therefore$ dividing by y^2, we get

$$y^2+\frac{1}{y^2}+\frac{p}{s^{\frac{1}{4}}}\cdot\left(y+\frac{1}{y}\right)+\frac{q}{s^{\frac{1}{2}}}=0.$$

$$\text{Make } y+\frac{1}{y}=u,$$

$$\therefore\ y^2+\frac{1}{y^2}=u^2-2,$$

$$\text{and } u^2+\frac{p}{s^{\frac{1}{4}}}u+\frac{q}{s^{\frac{1}{2}}}-2=0.$$

Whence u, $\therefore\ y+\frac{1}{y}$ is known. $\therefore\ y$ from solution of a quadratic, and thence the four values of x.

(3) Since $x^{12}-a^{12}=0$, $\therefore \left(\frac{x}{a}\right)^{12}=1=\cos 2p\pi \pm \sqrt{-1}.\sin 2p\pi$,

$$\therefore \frac{x}{a} = \cos\frac{2p}{12}\pi \pm \sqrt{-1}\sin\frac{2p}{12}\pi,$$

$$\therefore x=a\left(\cos\frac{p\pi}{6} \pm \sqrt{-1}.\sin\frac{p\pi}{6}\right),$$

and making $p=0, 1, 2, 3, 4, 5, 6$, we get

$$x = a, a\left(\cos\frac{\pi}{6} \pm \sqrt{-1}.\ \sin\frac{\pi}{6}\right), a\left(\cos\frac{\pi}{3} \pm \sqrt{-1}.\ \sin\frac{\pi}{3}\right)\cdots$$

$$a\left(\cos\frac{5\pi}{6} \pm \sqrt{-1}.\sin\frac{5\pi}{6}\right), -a,$$

or $=a, a(\cos 30 \pm \sqrt{-1}.\sin 30), a(\cos 60 \pm \sqrt{-1}.\sin 30)\ldots$

$$a\ (\cos 150 \pm \sqrt{-1}.\ \sin 150),\ -a,$$

or $= a, \frac{a}{2}(\sqrt{3} \pm \sqrt{-1}), \frac{a}{2}(1 \pm \sqrt{3})..\frac{a}{2}(-\sqrt{3}\pm\sqrt{-1}), -a.$

For the entire Theory of Binomial and Trinomial Equations, see *Private Tutor*, Alg. part ii. p. 399.

(4) The limiting equation has two of them. Hence

$$\left.\begin{array}{l} 3x^5-10x^3+15x+8=0 \\ \text{and } x^4-2x^2+1=0 \end{array}\right\}$$

have a common measure of the form $(x-a)^2=0$; and $\because$ $x^4-2x^2+1=(x^2-1)^2$; $\therefore$ either $(x-1)^2$, or $(x+1)^2$ is that common measure; that is, 1 or -1 is one of the equal roots.

By trial, -1 is found to be that root.

Dividing the proposed equation by $(x+1)^3$, or x^3+3x^2+3x+1, we get for the reduced quadratic

$$3x^2-9x+8=0,$$

whose roots are

$$\therefore\ \tfrac{3}{2} \pm \tfrac{1}{6}\sqrt{15}.\ \sqrt{-1}.$$

7. Since imaginary roots enter equations by pairs of the form $\alpha \pm \beta\sqrt{-1}$ (see *Private Tutor*, Alg. part ii. p. 46.) $\alpha-\beta\sqrt{-1}$ is also a root. $\therefore$ the given equation is divisible by

$$x^2-2\alpha x+\alpha^2+\beta^2.$$

Dividing and putting the quotient, which is of the form $x^2 + Px + Q$, equal to 0, and resolving the resulting equation, we find its roots to be

$$\tfrac{1}{2}\{-(p+2\alpha) \pm \sqrt{(p^2-4q-4\alpha p-8\alpha^2+4\beta^2)}\}.$$

But $4s = (\alpha^2+\beta^2)\ \{(p+2\alpha)^2-(p^2-4q-4\alpha p-8\alpha^2+4\beta^2)\}$ which gives the required transformation.

8. If $-\dfrac{r}{2} = a$ and $\sqrt{\left\{\left(\dfrac{r}{2}\right)^2 - \left(\dfrac{q}{3}\right)^3\right\}} = 6\sqrt{-1}$;

then, by *Cardan's* Rule,

$$x = \sqrt[3]{(a+b\sqrt{-1})} + \sqrt[3]{(a-b\sqrt{-1})}.$$

But (see third question),

$$a+b\sqrt{-1} = \sqrt{(a^2+b^2)}\ (\cos\theta + \sqrt{-1}.\ \sin\theta),$$

$$\text{in which } \cos\theta = \frac{a}{\sqrt{(a^2+b^2)}},$$

$$\therefore\ x = 2\ (a^2+b^2)^{\frac{1}{6}} \cos\frac{\theta}{3}.$$

$$\text{But } a^2+b^2 = \frac{r^2}{4} - \frac{r^2}{4} + \frac{q^3}{27} = \frac{q^3}{27},$$

$$\therefore\ x = 2.\ \left(\frac{q}{3}\right)^{\frac{1}{2}} \cos\frac{\theta}{3}, \text{ \&c.}$$

9. The reducing cubic is (*Wood*, art. 332.)

$$y^3 + 2qy^2 + (q^2 - 4s)\,y - r^2 = 0.$$

$$\therefore \text{ if } y = a,\ e = \sqrt{a}, \text{ and } f = \frac{q + e^2 - \dfrac{r}{e}}{2} = \frac{q + a - \dfrac{r}{\sqrt{a}}}{2}$$

$\therefore$ one of the component quadratics, is

$$x^2 + \sqrt{a}.\ x + \frac{q + a - \dfrac{r}{\sqrt{a}}}{2} = 0,$$

which being resolved gives

$$x + \frac{\sqrt{a}}{2} = \pm\sqrt{\left(\frac{a}{4} - \frac{q + a - \dfrac{r}{\sqrt{a}}}{2}\right)} =$$

$$\therefore\ x = -\frac{\sqrt{a}}{2} \pm \sqrt{\left(\frac{-q}{2} - \frac{a}{4} + \frac{r}{2\sqrt{a}}\right)},$$

and the other component quadratic gives

$$x = \frac{\sqrt{a}}{2} \pm \sqrt{\left(\frac{-q}{2} - \frac{a}{4} - \frac{r}{2\sqrt{a}}\right)}.$$

10. Divide the given equation by x^m, and the result is, collecting the extreme terms,

$$x^m + \frac{1}{x^m} + p\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + q\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \ldots = 0,$$

and making $x + \frac{1}{x} = 2\cos\theta$, then $x^2 + \frac{1}{x^2} = 2\cos 2\theta$, &c.

$\therefore$ the equation becomes

$$2\,\{\cos m\theta + p\cos(m-1)\theta + q\cos(m-2)\theta + \ldots\} = 0,$$

$$\text{or } \cos m\theta + p\cos(m-1)\theta + q\cos(m-2)\theta + \ldots\ldots = 0,$$

which equation is resoluble into one of the form

$$\cos^m\theta + \mathrm{P}\cos^{m-1}\theta + \ldots\ldots = 0,$$

an equation of m dimensions.

Otherwise.

Since the roots are in pairs of the form, $a, \frac{1}{a}$, the equation is decomposable into quadratic factors of the form $(x-a) \times \left(x - \frac{1}{a}\right)$, or of the form $x^2 - \left(a + \frac{1}{x}\right)x + 1$, that is,

$x^{2m} + px^{2m-1} + qx^{2m-2} + \text{\&c.} = (x^2 - \mathrm{A}x + 1)(x^2 - \mathrm{B}x + 1)$ &c. &c. see *Wood*, art. 325.

11. Make $a = \lambda\cos\theta$, $b = \lambda\sin\theta$; then

$$\lambda = \sqrt{(a^2+b^2)},\ \theta = \cos^{-1}\frac{a}{\sqrt{(a^2+b^2)}},$$

and $(a + b\sqrt{-1})^m + (a - b\sqrt{-1})^m = \lambda(\cos\theta + \sqrt{-1}\sin\theta)^m$
$+ \lambda(\cos\theta - \sqrt{-1}\sin\theta)^m$,

$= \lambda(\cos m\theta + \sqrt{-1}.\sin m\theta) + \lambda(\cos m\theta - \sqrt{-1}.\sin m\theta)$,

$= 2\lambda\cos m\theta$,

$= 2\sqrt{(a^2+b^2)}\cos m\theta$.

12. *Wood*, art. 338.

1	3	1, 3, −1, −3	3, 3, 3, −3
0	10	1, 2, 5, 10, −1, −2, −5, −10	2, −1, −1, −5
−1	−35	1, 5, 7, 35, −1, −5, −7, −35	1, −5, −1, −7

The decreasing progressions in which the common difference is a divisor of 10, are 3, 2, 1; 3, 1, -1; -3, -5, -7; consequently the factors to be tried are

$$2x+2,\ 2x+1,\ 2x-5,$$

of which $2x-5$ succeeds, and $\therefore \frac{5}{2}$ is a root giving the reduced quadratic.

13. Since

$$\frac{1}{(1-a)^2}=\frac{1}{1-2a+a^2}=1+2a+3a^2+\text{\&c. by division,}$$

$$\text{and}\qquad \frac{1}{1-a}=1+\ a+\ a^2+\text{\&c.}$$

$$\therefore\ \frac{1}{(1-a)^2}-\frac{1}{1-a}=a+2a^2+3a^3+\text{\&c.}$$

$$\text{similarly,}\ \frac{1}{(1-b)^2}-\frac{1}{1-b}=b+2b^2+3b^3+\text{\&c.}$$

&c.

$$\therefore\ s_1+2s_2+3s_3+\ldots\ \infty=\frac{1}{(1-a)^2}+\frac{1}{(1-b)^2}+\text{\&c.}$$
$$-\left(\frac{1}{1-a}+\frac{1}{1-b}+\text{\&c.}\right).$$

But, (*Wood*, art. 352.)

$$\frac{n+(n-1)p+(n-2)q+\ldots}{1+p+q+\ldots}=\frac{1}{1-a}+\frac{1}{1-b}+\text{\&c}$$

$$\therefore\left(\frac{n+(n-1)p+(n-2)q+\ldots}{1+p+q+\ldots}\right)^2=\frac{1}{(1-a)^2}+\frac{1}{(1-b)^2}+\text{\&c.}$$
$$+\,2\left\{\frac{1}{(1-a)(1-b)}+\frac{1}{(1-a)(1-c)}+\text{\&c.}\right\},$$

Also, $\because\ n\,x^{n-1}+(n-1)\,p\,x^{n-2}+(n-2)\,q\,x^{n-3}+\text{\&c.}$
$=(x-a)\,(x-b)\ldots(n-1)$ factors
$+\,(x-a)\,(x-c)\ldots(n-1)$ factors
$+$ &c.

$\therefore\ n\,x^n+(n-1)\,p\,x^{n-1}+\ldots$
$=x\,\{(x-a)\,(x-b)\ldots(n-1)$ factors $+$ &c.$\}$

Taking the Equation of Limits of this, we get

$$\begin{aligned} n^2x^{n-1}+(n-1)^2\,px^{n-2}+\ldots = {} & (x-a)\,(x-b)\ldots(n-1) \text{ factors} \\ & +\,(x-a)\,(x-c)\ldots(n-1) \text{ factors} \\ & \quad \&\text{c.} \\ & +\,2\,x\,\{(x-a)\,(x-b)\ldots(n-2) \text{ factors} \\ & \quad +\,(x-a)\,(x-c)\ldots(n-2) \text{ factors} \\ & \quad + \quad \&\text{c.}\} \end{aligned}$$

Hence, putting $x = 1$,

$$\frac{n^2+(n-1)^2p+(n-2)^2q+\ldots}{1+p+q+\ldots} = \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}+\&\text{c.}$$

$$+\;2\left\{\frac{1}{(1-a)\,(1-b)}+\frac{1}{(1-a)\,(1-c)}+\&\text{c.}\right\},$$

$$\therefore\; s_1+2\,s_2+3\,s_3+\ldots.\,\infty$$

$$=\left(\frac{n+(n-1)\,p+\&\text{c.}}{1+p+q+\&\text{c.}}\right)^2-\frac{n^2+(n-1)^2\,p+\&\text{c.}}{1+p+q+\&\text{c.}}.$$

It must be observed, as in the example of p. , that unless each of the roots be less than unit, this result will be absurd. For the left hand member will be ∞, whilst that on the right is finite.

TRINITY COLLEGE, 1826.

1. See *Private Tutor,* Alg. part ii. p. 15.

2. See *Private Tutor,* Alg. part ii. p. 46.

3. (1) *Private Tutor,* Alg. part ii. p. 112. for this and many other transformations of the same kind. *Ans.* p. .
   (2) See *Private Tutor,* ibid.
   Make $y = x\,(p-x) = p\,x - x^2$.
   (3) See *Private Tutor,* Alg. part ii. p. 272.

4. *Wood,* art. 314.

5. Let $x^n-p\,x^{n-1}+q\,x^{n-2}-\ldots=0$, be the given equation; then its Limiting Equation is

$$n\,x^{n-1}-(n-1)\,p\,x^{n-2}+(n-2)\,q\,x^{n-3}-\ldots=0;$$

also, let $a, b, c, \ldots; \alpha, \beta, \gamma, \ldots$ be their respective roots,

$$\frac{n\,x^{n-1}-(n-1)\,p\,x^{n-2}+\ldots}{x^n-p\,x^{n-1}+\ldots}=\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\text{\&c.}$$

then, $\because$ $x^n-p\,x^{n-1}+\ldots=(x-a)(x-b)(x-c)\ldots n$ factors

and $nx^{n-1}-(n-1)px^{n-2}+\ldots=n(x-\alpha)(x-\beta)\ldots(n-1)$ factors

the respective products P, P′, are

$\mathrm{P}=(\alpha-a)(\alpha-b)\ldots\times(\beta-a)(\beta-b)\ldots$ to $n\times(n-1)$ factors in all, and

$\mathrm{P}'=n^n(a-\alpha)(a-\beta)\ldots\times(b-\alpha)(b-\beta)\ldots$ to $(n-1)\times n$ factors,

whence it is evident, since $n\,(n-1)$ is even, that

$$\mathrm{P}:\mathrm{P}'::1:n^n.$$

6. (1) *Private Tutor*, Alg. part ii. p. 399.
   (2) See p. 70. The roots here are in Harmonic Progression.
   (3) This may be done by *Cardan's* Rule, by first taking away the second term, &c.; or, more briefly, thus,

$$x^3-2\,x^2+x-2=0,$$
$$\therefore\ x^3+x-2\,(x^2+1)=0,$$
$$\therefore\ (x^2+1)(x-2)=0,$$
$$\therefore\ \text{the roots are } 2,\ \pm\sqrt{-1}.$$

   (4) See p. 98. An erratum is, *for* -2 *read* $-r$.

7. See *Private Tutor*, Alg. part ii. p. 363.

8. Since, when $y=0$, $x=0$, make

$$y=\mathrm{A}\,x+\mathrm{B}\,x^2+\mathrm{C}\,x^3+\text{\&c.}$$
$$\frac{y^2}{2}=-\frac{\mathrm{A}^2x^2}{2}-\mathrm{AB}\,x^3+\text{\&c.}$$
$$\frac{y^3}{3}=\qquad\frac{\mathrm{A}^3x^3}{3}+\text{\&c.}$$
$$\therefore\ \mathrm{A}=1,\ \mathrm{B}-\frac{\mathrm{A}^2}{2}=0,\ \mathrm{C}-\mathrm{AB}+\frac{\mathrm{A}^3}{3}=0,$$
$$\therefore\ \mathrm{A}=1,\ \mathrm{B}=\frac{1}{2},\ \mathrm{C}=\frac{1}{2}-\frac{1}{3}=\frac{1}{2.\,3},\ \text{\&c.}$$
$$\therefore\ y=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\text{\&c.}=e^x-1.$$

See *Wood*, art. 346.

9. Generally, (*Wood*, art. 352.) if the equation be

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \&c. + L = 0,$$

$$S_m + p\,S_{m-1} + q\,S_{m-2} + \dots + m\,P = 0.$$

In this case, $p = 0$, $q = 0$, $r = 0$, &c. but $L = -1$.

Hence, when $m = n$, $S_n = n$, in all other cases $S_m = 0$.

10. See *Wood*, art. 509.

TRINITY COLLEGE, 1827.

1. *Wood*, art. 275.

2. See p. 90.

3. Let $y = m\,n\,p\,x$. Then substituting, &c.

$$\therefore\ y^3 - a\,n\,p\,y + b\,m^2\,n\,p^2\,y + c\,m^3\,n^3\,p^2 = 0,$$

is the required equation.

Secondly, make $y = p\,u$; then,

$$\therefore\ u^3 - 2\,u^2 - 33\,u + 14 = 0,$$

is the required equation.

4. *Wood*, art. 285.

5. *Wood*, art. 298, and 302.

6. First, transform the given equation in x to that in y, whose roots are the differences of the roots of the given equation; then find the Inferior Limit of the roots of the equation in y.

If $a, b, c,$ be the roots of the equation $x^3 - p\,x^2 + q\,x - r = 0$,

then, $(a-b)^2 = (a+b)^2 - 4\,a\,b$,

$\therefore$ if u be the character for the roots of the required equation, we have

$$u^2 = (p-x)^2 - \frac{4r}{x} = x^2 - 2\,p\,x + p^2 - \frac{4r}{x},$$

$$\left.\begin{array}{l} \therefore\ x^3 - 2px^2 + (p^2 - u^2)x - 4r = 0 \\ \text{But } x^3 - px^2 + qx - r = 0 \end{array}\right\}$$

$$\therefore p x^2 - (p^2 - u^2 - q)x + 3r = 0.$$

$$\text{Also}\ \left.\begin{array}{l} x^3 - 2px^2 + (p^2 - u^2)x - 4r = 0 \\ 4x^3 - 4px^2 + 4qx - 4r = 0 \end{array}\right\},$$

$$\therefore \left.\begin{array}{l} 3x^2 - 2px + 4q - p^2 + u^2 = 0 \\ px^2 - (p^2 - u^2 - q)x + 3r = 0 \end{array}\right\},$$

$$\left.\begin{array}{l} 3px^2 - 2p^2x + 4pq - p^3 + pu^2 = 0 \\ 3px^2 - (3p^2 - 3u^2 - 3q)x + 9r = 0 \end{array}\right\},$$

$$(p^2 - 3u^2 - 3q)x + 4pq - p^3 + pu^2 - 9r = 0,$$

$$\therefore x = \frac{pu^2 - p^3 + 4pq - 9r}{3u^2 - p^2 + 3q}.$$

Substituting this value of x in the equation

$$x^3 - px^2 + qx - r = 0$$

an equation of six dimensions will result, whose roots are

$$\pm(a-b),\ \pm(a-c)\ \pm(b-c).$$

If the equation be of the form $x^3 + qx - r = 0$; then the transformed equation is (*Private Tutor*, Alg. part ii. p. 300.)

$$u^6 + 6qu^4 + 9q^2u^2 + 4q^3 + 27r^2 = 0.$$

In the present instance, $q = -7$, $r = -7$,

$$\therefore\ u^3 - 42u^2 + 441u - 1372 + 1323 = 0,$$

of which an Inferior Limit may be found, as in p. 73.

7. Make $y = a + b + ab = p - x + \frac{r}{x}$; whence

$$xy = px - r^2 + r,$$

$$\left.\begin{array}{l} \text{or } x^2 + (y-p)x - r = 0 \\ \text{and } x^3 - px^2 + qx - r = 0 \end{array}\right\},$$

whence eliminating x, the equation in y will be that required. See *Private Tutor*, Alg. part ii. p. 235.

Otherwise.

Assume $y^3 - \mathrm{P}y^2 + \mathrm{Q}y - \mathrm{R} = 0$,

then $\mathrm{P} = 2(a+b+c) + ab + ac + bc$,

$= 2p + q$,

$\mathrm{Q} = (a+b+ab)(a+c+ac) + (a+b+ab)(b+c+bc)$
$+ (a+c+ac)(b+c+bc)$,

$= a^2 + (b+c)a + bc + (a+b)(ac+bc) + (a+c)bc$,

$$+b^2+(a+c)b+ac+ab(a+c)+a^2bc+ab(b+c)+ab^2c$$
$$+c^2+(a+b)c+ab+ac(b+c)+abc^2,$$
$$=a^2+b^2+c^2+3(ab+ac+bc)+a^2c+abc$$
$$+abc+b^2c+abc+bc^2$$
$$+a^2b+abc+a^2bc+ab^2+abc+ab^2c$$
$$+abc+ac^2+abc^2,$$
$$=p^2-2q+3q+3abc+a(ab+ac+bc)+b(ab+ac+bc)$$
$$+c(ab+ac+bc)+abc(a+b+c),$$
$$=p^2+q+3r+pq+pr,$$
$$\text{R}=(a+b+ab)(a+c+ac)(b+c+bc),$$
and multiplying these so as to group all those terms which are of the same dimension together, we have
$$\text{R}=a^2b^2c^2,$$
$$+abc^2(a+b)+a^2bc(b+c)+ab^2c(a+c),$$
$$+ac(a+b)(b+c)+bc(a+b)(a+c)+ab(a+c)(b+c)$$
$$+(a+b)(a+c)(b+c).$$
$$=r^2$$
$$+r(ac+bc+ab+ac+ab+bc)$$
$$+ac(ab+ac+b^2+bc)+bc(a^2+ac+ab+bc)$$
$$+ab(ab+ac+bc+c^2)$$
$$+(a^2+ac+ab+bc)(b+c),$$
$$=r^2+2qr,$$
$$+a^2b^2+a^2c^2+b^2c^2+3abc(a+b+c)$$
$$2abc+a^2(b+c)+b^2(a+c)+c^2(a+b),$$
$$=r^2+2qr+3pr+2r+a^2(p-a)+b^2(p-b)+c^2(p-c)$$
$$=r^2+2qr+3pr+2r+p(a^2+b^2+c^2)-(a^3+b^3+c^3).$$
But (*Wood*, art. 352.)
$$a^2+b^2+c^2=p^2-2q,$$
$$\text{and } a^3+b^3+c^3=p^3-3pq+3r,$$
$$\therefore \text{R}=r^2+2qr+3pr+2r+p^3-2pq-(p^3-3pq+3r)$$
$$=r^2+2qr+3pr+2r+pq-3r,$$
$=r^2+pq-r+3pr+2qr$, which gives the equation required.

8. *Wood*, art. 319. Also see p. 98.

9. See p. 78.

10. *Private Tutor,* Alg. vol. ii. part ii. p. 78.

11. *Private Tutor,* Alg. vol. ii. part. ii. p. 123.

12. *Wood,* art. 338.

As in *Wood,* it is found, that 3 and -5, are roots, but not 5. The other two roots are

$$\frac{3 \pm \sqrt{-11}}{2}.$$

13. *Private Tutor,* Alg. part ii. p. 399.

14. Since $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.\ 3} + \text{\&c.} = y$,

$$\therefore\ x = \log y,$$

and, it is well known, that $\log y$ cannot be expanded in a series according to the powers of y, for, if we assume

$$x = \text{A} + \text{B}y + \text{C}y^2 + \text{\&c.}$$

some of the coefficients will become infinite.

But, if we make

$$x = \text{A}\,(y-1) + \text{B}\,(y-1)^2 + \text{C}\,(y-1)^3 + \text{\&c.}$$

$$\text{then } \frac{x^2}{2} = \frac{\text{A}^2}{2}\,(y-1)^2 + \text{AB}\,(y-1)^3 + \text{\&c.}$$

$$\frac{x^3}{6} = \frac{\text{A}^3}{6}\,(y-1)^3 + \text{\&c.}$$

&c.

$$\therefore\ \text{A} = 1,\ \text{AB} + \frac{\text{A}^2}{2} = 0,\ \text{B} = -\tfrac{1}{2},$$

$$\text{C} = \tfrac{1}{2} - \tfrac{1}{6} = \tfrac{1}{3},\ \text{\&c.}$$

$$\therefore\ x = (y-1) - \tfrac{1}{2}\,(y-1)^2 + \tfrac{1}{3}\,(y-1^3 + \text{\&c.}$$

TRINITY COLLEGE, 1828.

1. *Wood,* art. 277.

2. *Private Tutor,* Alg. part ii. p. 15.

3. (1) Assume $y = a^2\,(b+c) = a^2\,(-p-a) = -x^3 - px$, and eliminate x from the equations,

$$\left.\begin{array}{l} x^3 + px^2 + qx + r = 0 \\ x^3 + px + y = 0 \end{array}\right\} \text{See } \textit{Private Tutor}, \text{Alg. pt. ii. p. 223.}$$

Otherwise; assume

$$y^3 - Py^2 + Qy - R = 0, \text{ then}$$

$P = a(ab+ac) + b(ab+bc) + c(ac+bc),$
$= a(ab+ac+bc) + b(ab+ac+bc) + c(ac+bc+ab) - 3abc,$
$= q(a+b+c) + 3r = -pq + 3r.$

Similarly, (see *Private Tutor*, Alg. part ii. p. 234.)
$Q = q^3 - 2pqr + 3r^2,$
$R = r^3 - pqr^2.$

(2) See *Private Tutor*, Alg. part ii. p. 300.

4. This can only be done by resolving a cubic. *Wood*, art. 285.

5. Let a, a^2, b, be the roots; then

(1) Since the last term is the product of all the roots with their signs changed, and the product of two of the roots is of the form $a \times a^2$, that factor which is a cube will give one of the roots, viz. a. For $a^3 = \pm 8$, $\therefore a = \pm 2$, and on trial we find $a = -2$, $\therefore a^2 = 4$, and the other root $= 0 + 2 - 4 = -2$.

(2) See p. .

(3) Divide by $4x^2$, and there results

$$\frac{x^2}{4} + p\frac{x}{2} + q + p.\frac{2}{x} + \frac{4}{x^2} = 0,$$

$$\text{or } \frac{x^2}{4} + \frac{4}{x^2} + p.\left(\frac{x}{2} + \frac{2}{x}\right) + q = 0,$$

$$\text{or } \left(\frac{x}{2} + \frac{2}{x}\right)^2 + p.\left(\frac{x}{2} + \frac{2}{x}\right) = 2 - q.$$

$$\therefore \frac{x}{2} + \frac{2}{x} = -\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} - q + 2\right)} = \text{P suppose.}$$

Hence we easily get x.

6. (1) *Wood*, art. 327.

(2) See *Private Tutor*, Alg. vol. ii. part ii. p. 71.

(3) *Idem, ibidem*, p. 78.

7. Reduce the equation to the form

$$x^3 + Qx + R = 0,$$

then make $x^2 = y + \phi + \phi' x$.

Eliminate x, and then assume, so as to determine ϕ, ϕ', &c. *Private Tutor*, Alg. part ii. p. 106.

8. See p. 79. Equation is

$$y^4 - \frac{mp}{s} y^3 + \left\{\frac{m^2}{s} + n(p^2 - 2s)\right\} y^2 - \frac{mn}{s} y + \frac{n^2}{s^2} = 0.$$

9. Let the equation be

$$x^n + p\,x^{n-1} + \ldots - M\,x^{n-m} - \ldots - P\,x^{n-\nu} \pm \&c. = 0.$$

Take the most unfavorable case in supposing all the coefficients from M negative and = P, and find such a value of x as shall always make $x^n - P\,(x^{n-m} + \ldots x^2 + x + 1)$ positive, and $\therefore$ *a fortiori*, the left hand member of the equation always positive; that is, find x such that x^n is always

$$> P.\frac{x^{n-m+1} - 1}{x - 1} > \frac{P\,x^{n-m+1}}{x-1} - \frac{P}{x-1}.$$

$$\text{Now } P\,\frac{x^{n-m+1}}{x-1} \text{ is } > \frac{P\,x^{n-m+1}}{x-1} - \frac{P}{x-1},$$

assume $\therefore x^n = \text{or} > \dfrac{P\,x^{n-m+1}}{x-1}$, or $\dfrac{x^{m-1}}{(x-1)^{m-1}}(x-1)^m = \text{or} > P$;

that is, make $(x - 1)^m = \text{or} > P$, or make $x = \text{or} > 1 + P^{\frac{1}{m}}$

$\therefore x^n - P\,(x^{m-n} + \ldots x^2 + x + 1)$ is always positive when $u = \text{or} > 1 + P^{\frac{1}{m}}$, and, *a fortiori*, $x^n + px^{n-1} + \ldots - M\,x^{n-m} - \&c. = 0$ is then always positive ; $\therefore 1 + P^{\frac{1}{m}}$ is a Superior Limit.

10. See p. 105.

TRINITY COLLEGE, 1829.

1. *Private Tutor*, Alg. part ii. p. 15.

2. See *Francœur's* Pure Mathematics.

3. See p. 110. Quest. 9.

4. Since the equation has equal roots, it has a common measure with the equation of limits. Hence the equal roots are 1, 1, 1, -3, -3, and the other root is -1.

5. By *Cardan's* Rule, or by the Method of Divisors, one of the roots is -2; the other two are $1 \pm \sqrt{-1}$.

6. *Wood*, art. 342.

7. *Wood*, art. 311.

8. *Wood*, art. 275.

9. *Wood*, art. 325.

10. Every Recurring Equation is decomposable into quadratic factors of the form $x^2 - mx + 1 = 0$. (*Wood*, art. 325.) If $a, \frac{1}{a}$ be the roots of this reduced recurring equation,

$$a = \frac{m}{2} + \sqrt{\left(\frac{m^2}{4} - 1\right)}, \text{ and } \frac{1}{a} = \frac{m}{2} - \sqrt{\left(\frac{m^2}{4} - 1\right)},$$

$\therefore \frac{1}{a}$ is not fractional; and similarly for the other pairs of reciprocal roots.

TRINITY COLLEGE, 1830.

1. *Private Tutor*, Alg. part ii. p. 106.

2. (1) By the Method of Divisors, one root is 6; the other roots are $\pm \sqrt{-3}$.

   (2) The equation is reciprocal, and one of its roots is -1. Dividing by $x + 1$ we get

$$x^4 - 22x^3 + 59x^2 - 22x + 1 = 0,$$

$$\therefore \left(x + \frac{1}{x}\right)^2 - 22\left(x + \frac{1}{x}\right) = -57,$$

$$\therefore x + \frac{1}{x} = 19 \text{ or } 3, \text{ and } \therefore x = \frac{19 \pm \sqrt{357}}{2}, \text{ or } \frac{3 \pm \sqrt{5}}{2},$$

3. *Private Tutor,* Alg. part ii. p. 46.

4. *Wood,* art. 271.

5. Find whether the Equation and its Limiting Equation have a common measure. (*Wood,* art. 319.)

6. *Private Tutor,* Alg. vol. ii. part ii. p. 78.

7. (1) *Wood,* art. 311.
   (2) There are two negative and three positive roots in the equation. *Wood,* art. 311.

8. *Wood,* art. 307, and 310.

9. *Private Tutor,* Alg. part ii. p. 399.

10. *Wood,* art. 352.

11. *Private Tutor,* Alg. part ii. vol. ii. p. 123.

ALGEBRA,

PART IV.

OR,

ANALYTICAL GEOMETRY.

TRINITY COLLEGE, 1830.

[P. 64.]

1. Let the given straight line be the axis of x, and α, β the co-ordinates of the given point, and x, y the co-ordinates of any point in the straight line, and θ the given angle.

$$y \sim \beta = (x \sim \alpha) \tan \theta.$$
$$= \tan \theta . x \sim \alpha . \tan \theta.$$

2. Let $y = \mathrm{A}x + \mathrm{B}$ be the equation required, in which A and B are to be determined; and, suppose

$$y' = ax' + b$$
$$y'' = a'x'' + b',$$

the equations of the given straight lines, in which a, b; a', b', are known. Also, let θ, θ' be the angles which the given lines make with x; and ϕ that made by the line $y = \mathrm{A}x + \mathrm{B}$. Then $\because$ it bisects the given lines, $\phi = \frac{1}{2}(\theta - \theta') + \theta' = \dfrac{\theta + \theta'}{2}$.

Now, at the point where all these straight lines meet, their co-ordinates are coincident; $\therefore$

$$\left.\begin{array}{l} y = \mathrm{A}x + \mathrm{B} \\ y = ax + b \\ y = a'x + b' \end{array}\right\} \begin{array}{l} (\mathrm{A} - a)x + (\mathrm{B} - b) = 0 \\ (a - a')x + b - b' = 0, \end{array}$$

and eliminating y and x, we get

$$A(b-b')-B(a-a') = a'b-ab' \quad \ldots\ldots (1)$$

$$\text{Also, } A = \tan\phi = \tan\frac{\theta+\theta'}{2},$$

$$= \frac{\tan\frac{\theta}{2} + \tan\frac{\theta'}{2}}{1-\tan\frac{\theta}{2}.\tan\frac{\theta'}{2}}.$$

But $\tan\theta = a$, $\tan\theta' = a'$

$$\therefore \tan\frac{\theta}{2} = \frac{-1+\sqrt{(1-a^2)}}{a}, \tan\frac{\theta'}{2} = \frac{-1+\sqrt{(1-a'^2)}}{2}$$

and, hence

$$A = \frac{a'\sqrt{(1+a^2)}+a\sqrt{(1+a'^2)}-(a+a')}{\sqrt{(1+a^2)}+\sqrt{(1+a'^2)}+aa'-\sqrt{(1+a^2)}\sqrt{(1+a'^2)}-1} \ldots (2)$$

which gives A ; and B is found from the form (1).

A more simple determination.

Take the origin of co-ordinates at the point of intersection; suppose the line $y = Ax + B$ to be drawn bisecting the $<$; take any point (x, y) in it; draw a $\perp$ through this point, meeting the lines $y' = ax' + b$, $y'' = a'x'' + b'$ in the points (x', y'), (x'', y''): then it may be easily shown, that

$$\left.\begin{aligned} 2x &= x' + x'' \\ 2y &= y' + y'' = ax' + a'x'' \\ x'^2 &+ (ax'+b)^2 = x''^2 + (a'x''+b')^2 \end{aligned}\right\}.$$

Find x' and x'' from the two first, and subtitute in the second, &c.; then resolve the resulting quadratic, and it will give the equation required.

3. Let a, b, c be the three sides of the $\triangle$ opposite to the $<^s$ A, B, C respectively. Also let α, α'; β, β'; γ, γ' be the co-ordinates of A, B, C.

Then, (area)^2 of $\triangle = \frac{a^2\,b^2}{4}.\ \sin^2 C$,

$$= \frac{a^2\,b^2}{4}\,(1-\cos{}^2 C) = \frac{a^2\,b^2}{4}.\ \left\{1 - \left(\frac{a^2+b^2-c^2}{2\,ab}\right)^2\right\},$$

$$= \tfrac{1}{4}\,\{4\,a^2\,b^2-(a^2+b^2-c^2)^2\}.$$

But $a^2 = (\beta-\gamma)^2 + (\beta'-\gamma')^2$,

$b^2 = (\alpha-\gamma)^2 + (\alpha'-\gamma')^2$,

$c^2 = (\alpha-\beta)^2 + (\alpha'-\beta')^2$.

Hence $(a^2+b^2-c^2)^2 = 4\,\{(\gamma-\alpha)\,(\gamma-\beta)-(\gamma'-\alpha')\,(\gamma'-\beta)\}^2$,

and $a^2b^2 = (\gamma-\alpha)^2.\ (\gamma-\beta)^2+(\gamma-\alpha)^2.\ (\gamma'-\beta')^2$

$$+\ (\gamma-\beta)^2\,(\gamma'-\alpha')^2+(\gamma'-\alpha')^2.\ (\gamma'-\beta')^2,$$

$\therefore$ (area)^2 of $\triangle = 4\,\{(\gamma-\alpha)^2.\ (\gamma'-\beta')^2$

$$+\ 2\,(\gamma-\alpha)\,(\gamma-\beta)\,(\gamma'-\alpha')\,(\gamma'-\beta') + (\gamma-\beta)^2\,(\gamma'-\alpha')^2\},$$

$\therefore$ area of $\triangle\ = 2\,\{(\gamma-\alpha)\,(\gamma'-\beta') + (\gamma-\beta)\,(\gamma'-\alpha')\}$.

This is the complete developement of the problem proposed and solved by Geometricus Emman. *Private Tutor*, and *Cambridge Mathematical Repository*, vol. ii. p. 96.

4. Let α, β; α', β'; α'', β'', &c. be the co-ordinates of the given points; x, y those of any point in the required locus. Then $\because$ the squares of the distances of the given points from the point x, y are respectively

$$(x-\alpha)^2 + (y-\beta)^2,$$
$$(x-\alpha')^2 + (y-\beta')^2,$$
$$(x-\alpha'')^2 + (y-\beta'')^2,$$

if n be the number of points, and c^2 the constant quantity, we have

$$x^2 + y^2 - \frac{2}{n}\,(\alpha+\alpha'+\alpha''+\ \ldots\ )\,x - \frac{2}{n}\,(\beta+\beta'+\beta''+\ldots)\,y +$$

$$\frac{\alpha^2 + \alpha'^2 + \alpha''^2 + \ldots + \beta^2 + \beta'^2 + \beta''^2 + \ldots}{n} = \frac{c^2}{n}\ \ldots\ (1)$$

whence the locus is a circle.

But if a, b be the co-ordinates of the centre of the circle, and R its radius; then

$$(x-a)^2 + (y-b)^2 = \mathrm{R}^2,$$

$$\therefore\ x^2+y^2-2ax-2by=\mathrm{R}^2-(a^2+b^2)\quad \ldots \quad (2)$$

Comparing this with (1),

$$a=\frac{a+a'+a''+\ldots}{n},$$

$$b=\frac{\beta+\beta'+\beta''+\ldots}{n},$$

$$\mathrm{R}^2=a^2+b^2+\frac{c^2}{n}-\frac{a^2+a'^2+\ldots+\beta^2+\beta'^2+\ldots}{n},$$

$$=\frac{c^2}{n}+\frac{(a+a'+..)^2+(\beta+\beta'+..)^2-n(a^2+a'^2+..+\beta^2+\beta'^2+..)}{n^2}$$

$$=\frac{c^2}{n}+\frac{(a^2+a'^2+..+\beta^2+\beta'^2+..)(1-n)+2(aa'+aa''+..+\beta\beta'+\beta\beta''+..)}{n^2}$$

which determine the co-ordinates, the centre, and the radius R of the circle.

5. Let a, β be the co-ordinates of the centre of the ⊙, and a its radius; then its rectangular equation is

$$(x-a)^2+(y-\beta)^2=a^2$$

But $x=r\cos\theta$, $y=r\sin\theta$; hence

$$r^2-2(a\cos\theta+\beta\sin\theta)\,r+a^2+\beta^2-a^2=0$$

the equation required.

6. Let the equations of the circles be

$$\left.\begin{array}{l}(x-a)^2+(y-\beta)^2=a^2\\(x'-a')^2+(y'-\beta')^2=a'^2\\(x''-a'')^2+(y''-\beta'')^2=a''^2\end{array}\right\}.$$

The conditions, that the two first, may respectively intersect the first and third, and second and third, are that

$$\sqrt{\{(a-a')^2+(\beta-\beta')^2\}}<a+a',$$
$$\sqrt{\{(a-a'')^2+(\beta-\beta'')^2\}}<a+a'',$$
$$\sqrt{\})a'-a'')^2+(\beta'-\beta'')^2\}}<a'+a''.$$

To find the points of intersection of the first two, make

$$x'=x,\ y'=y;\ \text{then}$$

$$\left.\begin{array}{l}(x^2-a)^2+(y-\beta)^2=a^2\\(x^2-a')^2+(y-\beta')^2=a'^2\end{array}\right\},$$

$$\therefore\ \left.\begin{array}{l}x^2+y^2-2ax+a^2-2\beta y+\beta^2=a^2\\x^2+y^2-2a'x+a'^2-2\beta'y+\beta'^2=a'^2\end{array}\right\}\ldots (a)$$

$$\therefore\ 2(a'-a)\,x+a^2-a'^2-2(\beta-\beta')\,y+\beta^2-\beta'^2=a^2-a'^2,$$

$$(a'-a)\,x+(\beta'-\beta)\,y=\frac{a^2-a'^2-(a^2+\beta^2)+(a'^2+\beta'^2)}{2}\quad\ldots(b).$$

Hence find x in terms of y, substitute in the first of equations (a); thence obtain two values of the form $y = \text{B} \pm \text{B}$. Similarly, substituting for y in terms of x, from the same equation we get two values of $x = \text{A} \pm \text{A}'$, suppose; then the co-ordinates of the points of intersection of the two first circles will be found.

Having thus obtained the points of intersection of the circles, we can easily obtain the equations of the straight lines passing through every two of them, and shall thence find they have one point common to them.

Otherwise.

The equation (b), being that of a straight line, can belong to no other than that which passes through the two points of intersection of the first two circles. Similarly, we find that

$$(a''-a)x_{,}+(\beta''-\beta)y_{,}=\frac{a^2-a''^2-(a^2+\beta^2)+(a''^2+\beta''^2)}{2}\ldots(c)$$

and

$$(a''-a')x_{,,}+(\beta''-\beta')y_{,,}=\frac{a'^2-a''^2-(a'^2+\beta'^2)+(a''^2+\beta''^2)}{2}\ldots(d)$$

are the two equations of the chords common to the first and third, and to the second and third, circles respectively.

Now, for brevity in the elimination, we may simplify these equations, by transferring the origin of co-ordinates to the centre of the first circle; that is, we may suppose

$$a=0,\ \beta=0.$$

They may be further simplified, by making the axis of x coincident with the line joining the centres of the two first

circles; that is, by making $\beta' = 0$; then our equations become

$$\left.\begin{aligned} a'x &= \frac{a^2 - a'^2 + a'^2}{2} \\ a''x_{\prime} + \beta'' y_{\prime} &= \frac{a^2 - a''^2 + a''^2 + \beta''^2}{2} \\ (a'' - a')\, x_{\prime\prime} + \beta'' y_{\prime\prime} &= \frac{a'^2 - a''^2 - a'^2 + a''^2 + \beta''^2}{2} \end{aligned}\right\} \ldots (e).$$

From the first of these it appears that the common chord of the two first circles is $\perp$ to the line joining their centres; the same is true of the other chords. To find the point of intersection of the first two chords, make $x_{\prime} = x$, $y_{\prime} = y$; then

$$\left.\begin{aligned} x &= \frac{a^2 - a'^2 + a'^2}{2\,a'} \\ \text{and } a''.\ \frac{a^2 - a'^2 + a'^2}{2\,a'} + \beta'' y &= \frac{a^2 - a''^2 + a''^2 + \beta''^2}{2} \end{aligned}\right\} \ldots (f).$$

Whence

$$\left.\begin{aligned} y &= \frac{a^2 (a' - a'') + a'^2 a'' - a''^2 a' + (a''^2 + \beta''^2)\, a' - a'^2 a''}{2\,a'\,\beta''} \\ \text{and } x &= \frac{a^2 - a'^2 + a'^2}{2\,a'} \end{aligned}\right\} \ldots (g)$$

which are the co-ordinates of the point of intersection of the common chords of the first and second circles, and first and third circles.

Again, to find the intersection of the first and third common chords, we must have $x = x_{\prime\prime}$ and $y = y_{\prime\prime}$. But then

$$\left.\begin{aligned} a'x &= \frac{a^2 - a'^2 + a'^2}{2} \\ (a'' - a')\, x + \beta'' y &= \frac{a'^2 - a''^2 - a'^2 + a''^2 + \beta''^2}{2} \end{aligned}\right\}.$$

Adding the first to the second, we get

$$\left.\begin{aligned} a''x + \beta'' y &= \frac{a^2 - a''^2 + a''^2 + \beta''^2}{2} \\ \text{and } a x &= \frac{a^2 - a'^2 + a'^2}{2} \end{aligned}\right\}$$

which, being the equations, as we before resolved, must give the same values of x and y. Whence the common chords all intersect in the same point determined by equations (g).

7. See *Hamilton's* Conics.

8. If a, b, be the semi-axes, and a', b', the semi-conjugate diameters, and θ be the inclination of the latter; then

(*Hamilton's* Conics, or *Francœur's* Pure Math.)

$$\left.\begin{aligned} a'^2 + b'^2 &= a^2 + b^2 \\ a'\,b' \sin\theta &= a\,b \end{aligned}\right\}.$$

Now, when $\theta = 90°$, $a'b' = a\,b$,

and then

$$a'^2 \pm 2\,a'\,b' + b'^2 = a^2 \pm 2\,a\,b + b^2,$$

$$\left.\begin{aligned} \text{or, } a' + b' &= a + b \\ \text{and } a' - b' &= a - b \end{aligned}\right\}, \text{ whence } a' = a, \text{ and } b' = b.$$

9. *Wood*, art. 508.

10. *Wood*, art. 510; and *Hamilton*, p. 191.

11. $$y = \frac{x^4 - a^2\,x^2 - b^2\,x^2 + a^2\,b^2}{c^3}.$$

Let $x = 0$; then $y = \dfrac{a^2\,b^2}{c^3}$,

$$x^4 - (a^2 + b^2)\,x^2 = c^3\,y - a^2\,b^2,$$

$$\therefore\ x^2 = \frac{a^2 + b^2}{2} \pm \sqrt{\frac{a^4 + b^4 - 2\,a^2\,b^2 + 4\,c^3\,y}{4}}.$$

Let $y = 0$, then

$$x^2 = \frac{a^2 + b^2}{2} \pm \frac{a^2 - b^2}{2} = a^2 \text{ or } b^2,$$

$$\therefore\ x = \pm\, a \text{ or } \pm\, b,$$

$\therefore$ the curve cuts the axis of y in one point only in the point

$$\left(x = 0,\ y = \frac{a^2\,b^2}{c^3}\right),$$

but it cuts the axis of x in the four points

$$(y = 0,\ x = \pm a),\ (y = 0,\ x = \pm b).$$

Make $x = \pm \infty$; then $y = \pm \infty$; $\therefore$ the curve has two infinite branches, one on each side of the origin of co-ordinates, but both on the same side of the axis of x.

The greatest negative value of y is $\dfrac{(a^2 + b^2)^2}{4\,c^3}$, and this gives $x = \pm \sqrt{\dfrac{a^2 + x^2}{2}}$.

The singular points, and other properties of the curve, are found by the Differential Calculus.

ARITHEMETIC AND ALGEBRA.

ST. JOHN'S COLLEGE, 1814.

[P. 65.]

1. Multiply by the least common multiple of the denominators, viz. 84, and there results

$$18\,x+42-8\,x+28+168+63=21\,x-84;\ \text{whence}$$
$$x=35.$$

2. The first reduces to

$$11\,x+14\,y=127\ \ldots\ (1)$$

Again, from the second

$$\frac{9\,xy-12\,x-9\,xy+110}{3\,y-4}=\frac{151-16\,x}{4\,y-1},$$

$$\text{or} -\frac{12\,x-110}{3\,y-4}=\frac{151-16\,x}{4\,y-1},$$

$$52\,x+13\,y=494,$$

$$4\,x+y=38\ \ldots\ \ (2)$$

The equations, when reduced, are

$$\left.\begin{array}{l} 11\,x+14\,y=127 \\ 4\,x+\quad y=\ \ 38 \end{array}\right\},$$

which give $x=9$, $y=2$.

3. This easily reduces to the quadratic

$$x^2-\frac{11}{5}x=\frac{12}{5},$$

$$\therefore x = \frac{11}{10} \pm \sqrt{\frac{361}{100}},$$

$$= \frac{11 \pm 19}{10} = 3, \text{ or } -\tfrac{4}{5}.$$

4. $$\frac{\sqrt{(x^2 - 15y - 14)}}{5} = \frac{x^2}{3} - 5y - 36,$$

$$= \frac{x^2 - 15y - 108}{3},$$

$$\therefore x^2 - 15y = \tfrac{3}{5}\sqrt{(x^2 - 15y - 14)} + 108,$$

$$\therefore x^2 - 15y - 14 - \tfrac{3}{5}\sqrt{(x^2 - 15y - 24)} = 94,$$

and solving this quadratic,

$$\therefore \sqrt{(x^2 - 15y - 14)} = \frac{3}{10} \pm \sqrt{\frac{9409}{100}},$$

$$= \frac{3}{10} \pm \frac{97}{10} = 10, \text{ and } -\frac{94}{10}, \text{ or } -\frac{47}{5},$$

$$\therefore x^3 - 15y - 14 = 100, \text{ or } \frac{47^2}{25},$$

$$\therefore x^2 - 15y = 114, \text{ or } \frac{2559}{25} \quad \ldots\ldots (1)$$

Again

$$\sqrt{\left(\frac{x^3}{3y} + \frac{x^2}{4}\right)} = \frac{x^2}{8y} + \frac{2x}{3} + \frac{y}{2},$$

$$\therefore \frac{x^3}{3y} + \frac{x^2}{4} = \frac{x^4}{64y^2} + \frac{4x^2}{9} + \frac{y^2}{4}$$

$$+ \frac{x^3}{6y} + \frac{x^2}{8} + \frac{2}{3}.\ xy,$$

$$\therefore \frac{x^4}{64y^2} - \frac{x^3}{6y} + \frac{23}{72}x^2 + \frac{2}{3}xy + \frac{y^2}{4} = 0.$$

The square root of this gives

$$\frac{x^2}{8y} - \frac{2x}{3} - \frac{y}{2} = 0, \quad \ldots\ldots (2)$$

$$\therefore 3x^2 - 16xy - 12y^2 = 0.$$

Divide this by x^2; then

$$12\frac{y^2}{x^2} + 16\frac{y}{x} = 3,$$

$$\frac{y^2}{x^2} + \frac{4}{3}\frac{y}{x} = \frac{1}{4},$$

$$\therefore \frac{y}{x}+\frac{2}{3} = \pm \sqrt{\left(\frac{4}{9}+\frac{1}{4}\right)} = \pm \frac{5}{6},$$

$$\therefore \frac{y}{x}=\frac{1}{6}, \text{ or} -\frac{3}{2} \quad . \ . \ . \ . \ (3)$$

By means of this equation and (1) we easily find all the simultaneous values of x and y; one pair of which is $x=12$, $y=2$.

5. Let x be the number of gallons required; then it is easily found that the clear profit on the ale is 17*l.* 10*s.*, or $17\frac{1}{2}l.$ the duty on the strong beer is $\frac{x}{40}l.$, that on the small beer is $\frac{50-x}{160}l.$ Hence the profit on the mixture is $30-\frac{50+3x}{160}$.

$\therefore$ by the question,

$$30-\frac{50+3x}{160} : 17\tfrac{1}{2} :: 10 : 7.$$

Hence, multiplying means and extremes,

$$210 - \frac{350+21x}{160} = 175,$$

whence

$$x = 250 \text{ gallons.}$$

6. Let x be the number of men at first in the garrison, and y that of the days the provisions lasted;
then, at the end of the sixth day, there were left $x-136$ men.

Also, as the provisions lasted $y-6$ days afterwards, 10 men dying daily, at last there were left only

$$x-136-(y-6)\,10,$$

$$\text{or } x-10y-76 \text{ men.}$$

But the stock of provisions at first being capable of sustaining the garrison for 8 days, allowing the loss of 6 daily, it consisted of

$$(2x-6\times 7)\,4 \text{ daily rations,}$$

and, at the end of six days, there had been consumed

$$(2x-6\times 5)\,3 \text{ rations.}$$

Consequently, by the question,

$$(2x-6\times 7)\,4-(2x-6\times 5)\,3=6\times 61,$$

$$\text{or } 4x-84-3x+45=183,$$

$$\therefore x=222 \text{ men at first.}$$

Again, the number of deaths in the first six days=136. Consequently 86 men had to subsist $y-6$ days upon 366 daily rations, 10 of them dying daily. Hence

$$\{(2\times 86-10\times(y-7)\} \times \frac{y-6}{2}=366,$$

$$\therefore (86-5y+35)(y-6)=366,$$

which quadratic being solved gives

$$y=12, \text{ or } \tfrac{91}{5},$$

of which the former only belongs to the problem.

x and y being found, the rest is evident.

7. Let x be the ratio of the quantities of gold and alloy in the guinea.

y the ratio of the values of equal weights of gold and alloy; then, if

w = weight of alloy in the guinea,

and v = value in shillings of one oz. of alloy,

then

$$w.\ v+x.\ w\times v.\ y=21s.,$$

and by the question

$$\tfrac{24}{25}wv+\tfrac{24}{25}.\ wx.\ vy.\ \tfrac{249}{239}=21s.$$

Hence, dividing the second by the first

$$\frac{\frac{24}{25}\left(1+\frac{249}{239}xy\right)}{1+xy}=1 \quad \ldots \quad (1)$$

which gives

$$xy=239 \quad \ldots\ldots \quad (2)$$

Again, by the question

$$x^2+y^2=11(x+y)+233\tfrac{9}{121} \quad \ldots \quad (3)$$

From (2) and (3) x and y are easily determined.

ST. JOHN'S COLLEGE, 1815.

1. First multiply by 12, and then reduce. Then multiply by 29, and collecting coefficients of x, &c., it will be found that

$$x = 72.$$

2. One term of the proportion of the second of the *simultaneous* equations is wanting. The student may make it what he pleases; then reduce the first equation by multiplying by 12, &c. &c.

3. First multiply by 10 and reduce; then multiply by $2x-5$, and a quadratic will be found, whose roots are

$$5 \text{ and } \tfrac{69}{10}.$$

4. From the first of these simultaneous equations, multiplying both numerator and denominator by

$$x+y+\sqrt{(x^2-y^2)},$$

we get

$$\frac{(x+y)^2+2(x+y)\sqrt{(x^2-y^2)}+x^2-y^2}{(x+y)^2-(x^2-y^2)} = \frac{9}{8y}(x+y),$$

Hence,

$$x+\sqrt{(x^2-y^2)} = \tfrac{9}{8}(x+y) \quad . \; . \; . \; . \quad (1).$$

Dividing by x, we have

$$\sqrt{\left(1-\frac{y^2}{x^2}\right)} = \frac{1}{8}+\frac{9}{8}\cdot\frac{y}{x}.$$

Hence,

$$\frac{y^2}{x^2}+\frac{18}{145}\cdot\frac{y}{x} = \frac{63}{145},$$

$$\therefore \frac{y}{x}+\frac{18}{290} = \pm\frac{192}{290},$$

$$\therefore \frac{y}{x} = \frac{174}{290}, \text{ and } -\frac{210}{290},$$

$$= \frac{3}{5} \text{ and } -\frac{21}{29} \quad . \; . \; . \; . \; . \; . \; . \quad (2).$$

Again, from the second of these simultaneous equations, we get

$$(x^2+y)^2-2x(x^2+y) = 506-x+y,$$
$$\therefore\ x^2+y-x = \pm\sqrt{(506+x^2+y-x)}$$
$$\therefore\ x^2+y-x+506 \mp \sqrt{(x^2+y-x+506)} = 506,$$
$$\therefore\ \sqrt{(x^2+y-x+506)} \mp \tfrac{1}{2} = \sqrt{(506+\tfrac{1}{4})} = \pm\sqrt{\tfrac{2025}{4}} = \pm\tfrac{45}{2},$$
$$\therefore\ \sqrt{(x^2+y-x+506)} = \pm\ 23.$$

$$\left.\begin{array}{ll}\text{Hence } & x^2+y-x=23\ \ .\ \ . \\ \text{But } & y=\tfrac{3}{5}x, \text{ or } -\tfrac{21}{29}x\end{array}\right\}\ .\ .\ .\ \ (3)$$

Hence all the simultaneous values of x and y may be found, of which one pair is $x=5$, $y=3$.

5. Let x be the number of soldiers in front,

then $40x$ = number of men in the mass,

and $4x$ = number of spectators.

Hence the number of men and spectators $= 44x$; and, by the question,

$$(x-100)\ 45 = 44x,$$
$$\therefore\ x = 4500 \text{ men}.$$

6. Let $2x$ be the number of persons, and £y the subscription of the youngest; then they subscribed respectively

$$y, y+5\ y+10\ .\ .\ .\ .\ y+5(2x-1).$$

Hence,

$$\{2y+5(2x-1)\}\ x=345\ .\ .\ .\ .\ (1).$$

Also

$$\{2y+5(x-1)\frac{x}{2} = 22x\ .\ .\ .\ .\ (2).$$

From (2)

$$2y+5x-5=44,$$
$$\therefore\ 2y+5x = 49\ .\ .\ .\ .\ (3)$$
$$\therefore\ \{49-5x+10x-5\}\ x=345,$$
$$\text{or } 44x+5x^2=345,$$
$$x^2+\tfrac{44}{5}x = 69,$$
$$x+\tfrac{22}{5} = \pm\tfrac{47}{5},\ \therefore\ x=5,$$
$$\therefore\ 2x = 10 \text{ and } y = 12l.$$

7. The geese went $\frac{5}{2}$ miles an hour.

Waggon $\frac{9}{4}$ miles an hour.

From the question, B set off $5 \div \frac{3}{2} = \frac{10}{3}$ hours after A.

Let x be the rate at which A and B travelled.

$\therefore$ when A was with the geese, the waggon was $2\,x + \frac{9}{2}$ miles from them, or from the 50th mile-stone. But, when A was at the 50th mile-stone, B was $x.\ \frac{10}{3}$ miles from the 50th mile-stone.

Now, the distance of waggon from where B meets it, when A was at the 50th mile-stone, is

$$2\,x + \tfrac{9}{2} - (50-31) + \tfrac{2}{3}\,x = \tfrac{8}{3}\,x - \tfrac{29}{2} \text{ miles,}$$

and the distance of B from that place is

$$\frac{10\,x}{3} + (50 - 31) - \tfrac{2}{3}\,x = \tfrac{8}{3}\,x + 19.$$

and their rates are as $\frac{9}{4}$, and x,

$$\therefore \tfrac{8}{3}\,x - \tfrac{29}{2} : \tfrac{8}{3}\,x + 19 :: \tfrac{9}{4} : x,$$

$$\therefore \frac{8x^2}{3} - \tfrac{29}{2}\,x = 6\,x + \tfrac{171}{4},$$

$$\text{or, } x^2 - \tfrac{123}{16}\,x = \tfrac{513}{32}.$$

Hence, $x = \dfrac{300.606}{.32}$ nearly $= 9.581$ miles an hour nearly.

$\therefore$ hence the answer is

$$\tfrac{10}{3} \times 9.581, \text{ or } 31.93 \text{ miles from London.}$$

ST. JOHN'S COLLEGE, 1816.

1. Multiply by 6; reduce; multiply by 17, &c.

Answer, $x = 51$.

2. Multiply by 21, and then by 20; then the first becomes

$$515\,x - 496\,y = -\,2634 \,\ldots\ldots (1)$$

Componendo,

$$30 : y - 2\,x + 15 :: \tfrac{5}{6} : \frac{y}{4} - \frac{x}{3} + \tfrac{1}{12},$$

$$:: 10 : 3\,y - 4\,x + 1.$$

Hence,

$$\left.\begin{array}{r} 5\,x - 4\,y = -6 \\ \text{But } 515\,x - 496\,y = -2634 \end{array}\right\}.$$

$$\text{Hence,}\quad x = 18,$$
$$y = 24.$$

3. First,

$$\frac{2\,x^2 - 6\,x + 2\,x^2 - 8\,x - 5\,x + 20}{x^2 - 7\,x + 12} = \frac{25}{3}.$$

$$\text{Hence,}\quad x^2 - \tfrac{118}{13}\,x = -\tfrac{240}{13},$$
$$\therefore\ x - \tfrac{59}{13} = \pm\sqrt{\tfrac{361}{132}} = \pm\tfrac{19}{13},$$
$$\therefore\ x = 6, \text{ and } \tfrac{40}{13}.$$

4. Clearing the first of fractions,

$$y^2 - 9\,x\sqrt{x} - 81 = 2\,x\sqrt{x}.\ y + 9\,x\sqrt{x},$$
$$\therefore\ y^2 - 2\,x\sqrt{x}.\ y + x^3 = 81 + 18\,x\sqrt{x} + x^3,$$
$$\therefore\ y - x\sqrt{x} = \pm\,(9 + x\sqrt{x}),$$
$$\therefore\ y = 9 + 2\,x\sqrt{x}, \text{ and } -9 \quad \ldots\ldots \quad (1).$$

Again, clearing the second of fractions,

$$y + 3\,x\sqrt{x} = 9 + y\,x\sqrt{x},$$
$$\text{or, } y\,(1 - x\sqrt{x}) = 9 - 3\,x\sqrt{x}. \quad \ldots\ldots \quad (2).$$

Hence the simultaneous values of x and y are

$$x = 0,\ y = 9;$$
$$x = \sqrt[3]{\tfrac{9}{4}},\ y = -9.$$

5. Let x be the rate of sailing with a fair wind,
y the distance between Dover and Calais.

Then, by the question,

$$2\,x = y. \quad \ldots\ldots \quad (1).$$

Again,

$$\text{time through first half} = \frac{y}{2} \div x = \frac{y}{2x},$$

$$\text{time } \ldots\ldots \text{ last half} = \frac{y}{2x} \div (x+2) = \frac{y}{2\,(x+2)},$$

$$\therefore \text{ whole time, the wind changing,} = \frac{y}{2x} + \frac{y}{2(x+2)}.$$

Also, time, wind not changing, $= \frac{y}{x}$;

$\therefore$ by the question,

$$\frac{y}{x} : \frac{y}{2x} + \frac{y}{2(x+2)} :: 7 : 6,$$

$$\left.\begin{aligned} \therefore \frac{6y}{x} &= \frac{7}{2}\cdot\frac{y}{x} + \frac{7}{2}\cdot\frac{y}{x+2} \ldots\ldots\ldots (2) \\ \text{But } y &= 2x \end{aligned}\right\}.$$

Hence, $x = 5$, and $y = 10$.

6. Let S be the junction of the sewer and river,
and C that of the streets,
D the point in CA where the drain is cut.

Also, let CA $= x$, CB $= y$, CS $= z$; then

$$SA = 6, \quad SB = z - 11,$$

and, from the $\triangle^s$ formed by these lines, we have

$$\frac{z-11}{y} = \frac{\sin(y, z)}{\sin(6, z)} = \frac{6}{x} \ldots (1)$$

$$\frac{z}{x-4} = \frac{\sin SDC}{\sin DSC} = \frac{6}{4} = \frac{3}{2}. \ (2).$$

Again, the cost of the sewer is zx.

cost of drain down both streets $= 9(x+y)$,

$\therefore$ by the question,

$$9(x+y) = zx + 54 \ldots\ldots (3).$$

The equations (1), (2), (3), reduce to

$$\left.\begin{aligned} xz - 11x &= 6y \\ 2z - 3x &= -12 \\ 9(x+y) - zx &= 54 \end{aligned}\right\}.$$

Add third to the first, and

$$9(x+y) - 11x - 6y = 54,$$

$$\text{or } 3y - 2x = 54 \ldots\ldots\ldots (4).$$

Also, from first and second,

$$3x^2 - 34x - 12y = 0 \ldots\ldots (5)$$

$$\therefore x^2 - 14x = 72,$$

which gives $x = 18$, and -4.

But, as the street leads to the river, x must be positive,

$$\therefore\ x = 18 \text{ chains} \ldots\ldots (6).$$

From (4),

$$y = 12 + 18 = 30 \text{ chains};$$

and, from (2),

$$z = \tfrac{3}{2}.(x-4) = 21 \text{ chains},$$

$$\text{whence, SB} = z - 11 = 10 \text{ chains}.$$

7. Let $x =$ the number of the family,
$y =$ the sum in pence saved by the man a month,
$z =$ the common difference of their monthly savings,
and $w =$ the price of wheat per bushel,

then y, $y-z$, $y-2z$, ... $y-(x-1)z$ are the respective savings.

Now the seventh child saved $y-(9-1)z$, or $y-8z$ pence,
and the eldest saved $y-(3-1)z$, or $y-2z$ pence,
and the 5th saved $y-(7-1)z$, or $y-6z$ pence,

and $\therefore\ y+y-z+y-2z\ldots+y-(x-1)z=\frac{1}{6}.(y-8z)w-39$

$$\text{or } \{2y-(x-1)z\}\frac{x}{2} = \tfrac{1}{6}(y-8z)w-39 \ldots. (1).$$

$$\text{Also } w = \frac{2y-8z+120}{2}, \text{ or } y-4z+60 \ldots (2).$$

Thirdly, the two youngest dying, there are left $(x-2)$ in the family,

$$\therefore (y-2z-12)(x-2) = \{2y-(x-1)z\}\frac{x}{2} - \frac{5\times 21\times 12}{12} (3)$$

it being assumed that the months are calendar,

$$\text{and } (y-2z-2)(x-2) = \{2y-(x-1)z\}\cdot\frac{x}{2} + 25 \ldots (4)$$

these four equations enable us to find x, y, z, w.

Subtracting the (3) from (4), we get

$$10(x-2) = 130,$$

$$\therefore\ x = 15.$$

Hence, by substitution, we get

$$\left.\begin{array}{l}(2y-14z)\frac{15}{2} = \frac{1}{6}(y-8z)w-39 \\ w=y-4z+60 \\ \text{and } (y-2z-2)\times 13 = (2y-14z)\times\frac{15}{2}+25\end{array}\right\},$$

$$\left.\begin{array}{l}y^2-30y-12zy+32z^2+150z-234=0 \\ \text{and } 2y+79z+51=0\end{array}\right\},$$

from which, first eliminating y, z and y may be found, and thence w may be found from equation (2).

ST. JOHN'S COLLEGE, 1817.

1. Multiply by 10. Ans. $x = 3$.

2. From the first $\sqrt{(y-x)}+\sqrt{(a-x)}=\sqrt{y}$.
Hence the second becomes

$$\sqrt{y} : \sqrt{(a-x)} :: 5 : 2,$$

$$\therefore\ y = \frac{25}{4}(a-x).$$

Hence, by substitution,

$$\left.\begin{array}{l}x = \dfrac{4a}{5} \\ y = \dfrac{5}{4}a\end{array}\right\}.$$

3. Clearing it of fractions, we get

$$= x - \tfrac{5}{3}\sqrt{x} = \tfrac{2}{3},$$

$$\sqrt{x} = \tfrac{5}{6} \pm \sqrt{\left(\frac{25}{36}+\frac{24}{36}\right)} = \frac{5\pm 7}{6} = 2, \text{ and } -\tfrac{1}{3},$$

$$\therefore\ x = 4, \text{ and } \tfrac{1}{9}.$$

4. From the second equation,

$$\left(\frac{y^2}{2}\right)^2 - 2x.\left(\frac{y^2}{2}\right) = -1,$$

which gives

$$y^2 = 2x \pm 2\sqrt{(x^2-1)},$$

$$\left.\begin{array}{l}\therefore\ y^2 - 2\sqrt{(x^2-1)} = 2x \\ \text{or } y^2 - 2\sqrt{(x^2-1)} = 2x-4\sqrt{(x^2-1)}\end{array}\right\}\ \ldots\ (1)$$

Hence the first equation gives

$$\frac{y-2\sqrt{(x+1)}}{2x}+\frac{y-\sqrt{(x-1)}}{2x}=0,$$

$$\therefore\ y\quad 2\sqrt{(x+1)}+y-\sqrt{(x-1)}=0,$$

$$\therefore\ 2y=2\sqrt{(x+1)}+\sqrt{(x-1)}\ \ldots\ (2).$$

Again,

$$\left.\begin{aligned} 4y^2 &= 5x+3+4\sqrt{(x^2-1)}\\ \text{But }4y^2 &= 8x+8\sqrt{(x^2-1)}\end{aligned}\right\},$$

$$\therefore\ 3x+4\sqrt{(x^2-1)}=3,$$

$$3(x-1)+4\sqrt{(x^2-1)}=0,$$

$$\therefore\ 9x^2-18x+9=16x^2-16,$$

$$\therefore\ 7x^2+18x=25,$$

$$x^2+\tfrac{18}{7}x=\tfrac{25}{7},$$

$$\therefore\ x+\frac{9}{7}=\pm\sqrt{\left(\frac{81}{49}+\frac{175}{49}\right)}=\pm\sqrt{\frac{256}{49}}=\pm\frac{16}{7},$$

$$\therefore\ x=1,\text{ and }-\tfrac{25}{7}.$$

$$\text{Hence, }y=\sqrt{2},\text{ and }\sqrt{-\tfrac{18}{7}}+\sqrt{-\tfrac{8}{7}},$$

$$=\sqrt{2},\text{ and }\sqrt{-\tfrac{18}{7}}+\tfrac{1}{2}\sqrt{-\tfrac{32}{7}}.$$

Two other pairs of simultaneous values of x and y may be found by using the second of equations (1).

5. Let x be the number of bushels,
and y the shillings per bushel he gave for it.
Then xy is the cost.

By the question, (the interest for 6 months being $\frac{xy}{40}s.$)

$$x(y-1)=xy+\frac{xy}{40}-5y,$$

$$\text{or }xy+40x-200y=0\ \ldots\ (1).$$

Again, his expected gain was $3x$,

the interest for the year was $\frac{xy}{20}s.$

$\therefore$ by the question,

$$x(y-2)=xy+\frac{xy}{20}-(3x-10),$$

$$\text{or }20x-xy=200\ \ldots\ (2).$$

$$\text{Hence } y = \frac{20\,x - 200}{x},$$

$$\therefore\ 20\,x - 200 + 40\,x - 200.\ \frac{20\,x - 200}{x} = 0,$$

$$\text{or } x - 10 + 2\,x - 10.\ 20 + \frac{2000}{x} = 0,$$

$$3\,x^2 - 210\,x = -\,2000,$$

$$\therefore\ x^2 - 70x = -\,\frac{2000}{3} = -\,\frac{6000}{9},$$

$$x - 35 = \pm\sqrt{\left(1225 - \frac{6000}{9}\right)} = \pm\,\frac{1}{3}\sqrt{5025},$$

$$= \pm\,\frac{70.\ 88}{3} = \pm\ 23.\ 62.$$

$$\therefore\ x = 58.\ 62 \text{ bushels, or } 11.\ 38 \text{ bushels},$$

$$\therefore\ y = \frac{20\,x - 200}{x} = \frac{20\,(x - 10)}{x},$$

$$= \frac{20 \times 48.\ 62}{58.\ 62} = \frac{486.\ 2}{29.\ 31} = 16s.\ 7d.\ \frac{57}{977},$$

$$\text{Also, } y = \frac{(11.\ 38 - 10)\ 10}{5.\ 69} = 2s.\ 5d.\ \frac{59}{569}.$$

6. Let x be the days of the expected passage,

y the number of men alive;

then the number of deaths $= (x + 7 \times 3 - 30) \times 3$,

$$= 3\,x - 27,$$

$$\therefore\ y = 175 - (3\,x - 27),$$

$$\text{or } 3\,x + y = 202\ \ldots\ldots\ (1).$$

Again, if w be the water allowed each man per diem,

$175 \times x \times w =$ whole store of water.

Also,

$175 \times 30 \times w =$ water drunk during the first 30 days, and $w\ \{172 + 169 + 166\ \ldots\ldots\ (x + 21 - 30) \text{ terms}\} =$ water drunk during the rest of the voyage; which, being an arithmetic series whose common difference is $-\,3$, &c. is

$$w\ \{2 \times 172 - (x - 10)\,3\}\ \frac{x - 9}{2};$$

$\therefore$ by the question,

$$w\,(374 - 3\,x)\,\frac{x-9}{2} + 175 \times 30\,w = 175\,x\,w,$$

$$\text{or }\;(374 - 3\,x)\,(x-9) + 10500 = 350\,x\ldots(2).$$

This equation reduced, becomes

$$\therefore\; x^2 - 17\,x = 2378,$$

$$\text{whence }\; x = 58, \text{ and } - 41,$$

the former of which only applies to the question.

Hence $y = 202 - 3\,x = 202 - 174 = 28$.

7. Let x be the gallons at first in the hold,

and y the gallons per hour of influx,

z the gallons pumped out by A in an hour;

then $\frac{2}{3}\,z \times \frac{5}{4}$ or $\frac{5}{6}\,z$ . . . . B . . . . . .

Now, if t be the time of A emptying the hold; then

$$t\,y = \text{influx during that time},$$

$$\text{and } \therefore\; \frac{t\,y + x}{z} = t,$$

$$\therefore\; t = \frac{x}{z-y},$$

$\therefore$ water thrown out by $\mathrm{B} = \dfrac{x}{z-y} \times \frac{5}{6}\,z$,

$$\text{that by A} = \left(13\tfrac{1}{3} - \frac{x}{z-y}\right) z,$$

and the whole water pumped was $x + 13\frac{1}{3}\,y$,

$\therefore$ by the question,

$$\left.\begin{aligned}
&\tfrac{5}{6}\cdot\frac{x\,z}{z-y} + \left(\tfrac{40}{3} - \frac{x}{z-y}\right) z = x + \tfrac{40}{3}\,y\,..(1)\\
&\text{Again,}\\
&(z + \tfrac{5}{6}\,z) \times (3 + \tfrac{45}{60}) = x + (3 + \tfrac{3}{4})\,y \ldots\ldots\ (2)\\
&\text{also, } z\,(3 + \tfrac{3}{4}) = \left(\tfrac{40}{3} - \frac{x}{z-y}\right) z + 100 \ldots\ldots\ldots\ (3)
\end{aligned}\right\}$$

These equations reduced, become

$$\left.\begin{array}{l} \frac{40}{3}z - \frac{1}{6}\dfrac{xz}{z-y} = x + \frac{40}{3}y \\ \frac{55}{8}z = x + \frac{15}{4}y \\ \dfrac{xz}{z-y} - \frac{115}{12}z = 100, \end{array}\right\}$$

$$\left.\begin{array}{l} 80\,(z-y)^2 - 6\,x\,(z-y) - x\,z = 0 \\ 30\,(z-y) + 25\,z - 8\,x = 0 \\ 12\,x\,z - (115\,z + 1200)\,(z-y) = 0 \end{array}\right\}.$$

To simplify these, make $z - y = u$; then

$$\left.\begin{array}{l} 80\,u^2 - 6\,x\,u - x\,z = 0 \\ 30\,u + 25\,z - 8\,x = 0 \\ 12\,x\,z - (115\,z + 1200)\,u = 0 \end{array}\right\}.$$

From the second,

$$u = \frac{8\,x - 25\,z}{30}.$$

Hence, substituting in the first,

$$\frac{4}{45}\cdot(8\,x - 25\,z)^2 - \frac{8\,x^2 - 25\,x\,z}{5} - x\,z = 0,$$

which is reducible to

$$46\,x^2 - 355\,x\,z + 625\,z^2 = 0 \ldots\ldots\ldots\ldots (4)$$

Again, substituting for u in the third,

$$12\,x\,z - \frac{23\,z + 240}{6}.\,(8\,x - 25\,z) = 0,$$

$$575\,z^2 - 112\,x\,z + 6000\,z - 1920\,x = 0 \ldots (5)$$

From (4), dividing by x^2, we have

$$625\,\frac{z^2}{x^2} - 355\,\frac{z}{x} + 46 = 0,$$

$$\left(\frac{z}{x}\right)^2 - \frac{71}{125}\left(\frac{z}{x}\right) = -\frac{46}{625},$$

$$\therefore \frac{z}{x} - \frac{71}{250} = \pm\sqrt{\left(\frac{71^2}{250^2} - \frac{46}{625}\right)} = \pm\frac{21}{250},$$

$$\therefore \frac{z}{x} = \frac{71 \pm 21}{250} = \frac{92}{250}, \text{and } \frac{50}{250} = \frac{46}{125}, \text{and } \frac{1}{5},$$

$$\therefore x = \frac{125}{46}\,z, \text{ or } 5\,z \ldots\ldots\ldots\ldots (6)$$

Substituting the latter value in (5),

$$575 z^2 - 112 \times 5 z^2 + 6000 z - 1920 \times 5 z = 0,$$

$$\therefore 115 z^2 - 112 z^2 + 1200 z - 1920 z = 0,$$

$$\text{or } 3 z^2 = 720 z,$$

$$z^2 = 240 z,$$

$$\therefore z = 0, \text{ or } 240,$$

of which the latter, viz. 240 gallons, alone answers the question.

$$\text{Hence, } x = 5 z = 1200 \text{ gallons},$$

$$\text{and } \because \tfrac{55}{8} z = x + \tfrac{15}{4} y,$$

$$\therefore \tfrac{55}{8} z = 5 z + \tfrac{15}{4} y,$$

$$\therefore 11 z = 8 z + 6 y,$$

$$\therefore 3 z = 6 y,$$

$$\text{and } y = \frac{z}{2} = 120 \text{ gallons}$$

that is, x, y and z are respectively

1200, 120, 240 gallons.

Again, if we use $x = \frac{125}{46} z$; substituting in (5), we get

$$z = 0, \text{ or } - \tfrac{240}{83},$$

neither of which answer the conditions of the question.

ST. JOHN'S COLLEGE, 1818.

1. First multiply by 24; collect like terms of the result; then multiply by 7, &c. Ans. $x = 7$.

2. Multiply the first by 40; collect like terms of the result; then multiply by 21, and collect like terms.

Again, multiply the second by 18; collect like terms of the result. Then eliminate x from the two new equations, and the resulting equation in y, will give

$$y = -\frac{298398}{39757},$$

whence x is easily found.

3. This equation is

$$x^{\frac{7}{3}} + 41\ x^{-\frac{2}{3}} = 97 x^{-\frac{2}{3}} + x^{\frac{5}{6}},$$

$$\therefore\ x^{\frac{7}{3}} = 56\, x^{-\frac{2}{3}} + x^{\frac{5}{6}},$$

Multiplying by $x^{\frac{2}{3}}$, we get,

$$x^3 - x^{\frac{3}{2}} = 56,$$

and solving the quadratic,

$$x^{\frac{3}{2}} - \tfrac{1}{2} = \pm \sqrt{\tfrac{225}{4}} = \pm \tfrac{15}{2},$$

$$\therefore\ x^{\frac{3}{2}} = 8, \text{ or } -7,$$

$$\therefore\ x = 4, \text{ or } \sqrt[3]{49}.$$

4. From the first,

$$x^2y^2 - 80\, y^2 - 272 = -\, y^2 \sqrt{\left(x^2 - \frac{272}{y^2}\right)},$$

$$\therefore \left(x^2 - \frac{272}{y^2}\right) + \sqrt{\left(x^2 - \frac{272}{y^2}\right)} = 80,$$

and solving the quadratic,

$$\sqrt{\left(x^2 - \frac{272}{y^2}\right)} + \tfrac{1}{2} = \pm \sqrt{(\tfrac{1}{4} + 80)} = \pm \tfrac{1}{2} \sqrt{321},$$

$$\therefore\ x^2 - \frac{272}{y^2} = \tfrac{1}{4} \mp \tfrac{1}{2} \sqrt{321} + \frac{321}{4},$$

$$= \frac{161 \mp \sqrt{321}}{2} \quad \ldots\ldots (1).$$

Again, from the second,

$$x^2 - \frac{36}{y^2} = 35\, \frac{x}{y},$$

$$\therefore\ x^2 y^2 - 35\, xy = 36,$$

and solving the quadratic,

$$xy = \frac{35}{2} \pm \sqrt{\frac{35^2 + 144}{4}} = \frac{35 \pm 37}{2} = 36, \text{ or } -1 \ldots (2).$$

But from (1),

$$x^2y^2 - 272 = \frac{161 \mp \sqrt{321}}{2}\, y^2,$$

making $xy = 36$,

$$y^2 = 1024 \times \frac{2}{161 \mp \sqrt{321}} = \frac{1024}{6400} \cdot \frac{161 \pm \sqrt{321}}{2},$$

$$\therefore y = \pm \tfrac{1}{5} \sqrt{(322 \pm 2\sqrt{321})},$$

$$= \pm \tfrac{1}{5} \left\{ \sqrt{\frac{322 + 320}{2}} \pm \sqrt{\frac{322 - 320}{2}} \right\},$$

$$= \pm \tfrac{1}{5} (\sqrt{321} \pm 1),$$

$$\left.\therefore x = \frac{36}{y} = \pm \frac{36 \times 5}{\sqrt{321} \pm 1} = \pm \frac{9}{16} \cdot (\sqrt{321} \mp 1)\right\} . (3).$$

Again, making $xy = -1$, and substituting in (1), we get

$$1 - 272 = \frac{161 \mp \sqrt{321}}{2} y^2,$$

$$\therefore y^2 = \frac{271 \times 2}{161 \mp \sqrt{321}} \cdot -1,$$

$$= \frac{271}{6400} \times \frac{161 \pm \sqrt{321}}{2} \times -1,$$

$$\therefore y = \pm \frac{\sqrt{271}}{\sqrt{25600}} \cdot \sqrt{(322 + \sqrt{321})} . \sqrt{-1},$$

$$= \pm \frac{\sqrt{271}}{160} \cdot (\sqrt{321} \pm 1) . \sqrt{-1},$$

$$\text{and } x = \frac{-1}{y} = \mp \frac{160}{\sqrt{271}} \cdot \frac{1}{(\sqrt{321} \pm 1)\sqrt{-1}},$$

$$= \pm \frac{160}{\sqrt{271}} \cdot \frac{(\sqrt{321} \mp 1)\sqrt{-1}}{320 \times -1},$$

$$= \pm \frac{1}{2\sqrt{271}} \cdot (\sqrt{321} \mp 1) . \sqrt{-1}.$$

5. Let x be the age of A, January 1, 1799; then the donations amounted to

$x + (x + 1) + (x + 2) \ldots$ to 8 terms $= (2x + 7) \times 4$ groats, and by the question,

$$(2x + 7)\, 4 = 7 \times 20 \times 3 + 18 \times 3 + 2 = 476,$$

which gives $x = 56$,

$\therefore$ the year in which he was born $= 1799 - 56 = 1743$,

and A's age at his death $= 56 + 7 = 63$.

6. Let x be the price of a share in the canal speculation. The profits of the canal speculation $= 5x + 595l.$,

$$\therefore \text{ the gain of each} = \frac{5x+595}{15} = \frac{x+119}{3}l.$$

$$\therefore \text{ each ventured on the steam boats } \frac{x+119}{3} - 173 = \frac{x-400}{3}l.$$

$$\text{Hence the total advance} = \frac{8x-3200}{3}l.$$

$$\text{Hence the total loss} = \frac{8x-3200}{3} + \frac{8x+952}{3} + 368,$$

$$= \frac{16x-1144}{3},$$

$$\therefore \text{ A's loss} = \frac{2x-143}{3} = 419 \text{ by the question,}$$

$$\therefore x = 700l.,$$

$\therefore$ the price of a share in the steam boat speculation is

$$\frac{x-400}{3} = 100l.$$

7. Let x be the width of each of the two large warehouses built by A and B,

y . . . . small . . . . . .

The contents of A's warehouses are x^3 and y^3.

Hence, by the question,

$$x^3+y^3-xy^2-yx^2 = 73728 \quad . \; . \; . \; . \; . \; (1).$$

Again,

$$x^2 - y^2 = \text{base of C's,}$$

and, by the question,

$$x^2-y^2 = 2688 - 8\sqrt{(x^2-y^2)}. \; . \; . \; . \; (2).$$

From the second, solved as a quadratic,

$$\sqrt{(x^2-y^2)} = -4 \pm 52,$$

$$= 48, \text{ and } -56,$$

$$\therefore x^2-y^2 = 48^2, \text{ and } 56^2 = 2304 \text{ and } 3136. \; . \; (3).$$

From the equation (1), we have

$$x^3+y^3-xy(x+y) = 73728,$$

$$\text{or } (x^2-xy+y^2)(x+y)-xy(x+y) = 73728,$$

$$\therefore (x-y)^2(x+y) = 73728,$$

$$\text{that is, } (x^2-y^2)(x-y) = 73728 \; . \; . \; . \; . \; . \; . \; . \; . \; (4).$$

Dividing this by the equation (3),

$$x-y = \frac{73728}{2304} \text{ and } \frac{73728}{3136},$$
$$= 32 \text{ and } \frac{1152}{49}. \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Dividing equation (3) by equation (5),

$$x+y = \frac{2304}{32} \text{ and } \frac{3136 \times 49}{1152},$$
$$= 72 \text{ and } \frac{49^2}{18} = 72 \text{ and } \frac{2401}{18},$$
$$\left.\begin{aligned} \therefore\ x+y &= 72 \text{ and } \frac{2401}{18} \\ x-y &= 32 \text{ and } \frac{1152}{49} \end{aligned}\right\},$$
$$\therefore\ x = 52,\ y = 20,$$
$$\text{also, } x = \frac{138385}{1764},\ y = \frac{966913}{1764}.$$

Also, the width of C's $= \sqrt{(x^2 - y^2)} = 48$ in the one case, and is easily also found for the other values of x, y.

ST. JOHN'S COLLEGE, 1823.

1. Clear it of fractions, by multiplying it by 12, &c.

Ans. $x = 6$.

2. Eliminate z by means of the first and second; then eliminate z from the second and third. From the two new equations eliminate y, and there will result an equation in x only. This gives $x = 1$; whence $y = 2$, and $z = 4$.

Otherwise.

Multiply the first by ϕ, and the second by ϕ', and add the third to the results; the sum is

$$(\phi + 2\phi' + 3)\,x + (2\phi + 3\phi' + 1)\,y + (3\phi + \phi' + 2)\,z = 17\phi + 12\phi' + 13.$$

To eliminate y and z, assume

$$\left.\begin{aligned} 2\phi + 3\phi' + 1 &= 0 \\ 3\phi + \phi' + 2 &= 0 \end{aligned}\right\},$$

$$\text{Hence, } \phi = -\tfrac{5}{7},$$

$$\text{and } \phi' = \tfrac{1}{7}.$$

Consequently, $(-\frac{5}{7} + \frac{2}{7} + 3)\,x = -\frac{85}{7} + \frac{12}{7} + 13$,

$$\therefore \tfrac{18}{7}\,x = \tfrac{18}{7}, \quad \therefore x = 1.$$

Similarly, y and z may be found, independently of the value of x; or, if two of the equations be reduced, they may thence be found.

3. Divide the equation by $x^{\frac{1}{m}}$, and there results

$$a^2 b^2\, x^{\frac{m-n}{mn}} - 4\,(ab)^{\frac{3}{2}} x^{\frac{m-n}{2mn}} = (a-b)^2,$$

$$\therefore\ x^{\frac{m-n}{mn}} - \frac{4}{\sqrt{(ab)}}\, x^{\frac{m-n}{2mn}} = \frac{(a-b)^2}{a^2 b^2}.$$

Solving this as a quadratic,

$$\begin{aligned} x^{\frac{m-n}{2mn}} &= \frac{2}{\sqrt{(ab)}} \pm \sqrt{\left\{\frac{4}{ab} + \frac{(a-b)^2}{a^2 b^2}\right\}}, \\ &= \frac{2}{\sqrt{(ab)}} \pm \frac{a+b}{ab} = \frac{2\sqrt{(ab)} \pm (a+b)}{ab}, \\ &= \pm \frac{\{a+b \mp 2\sqrt{(ab)}\}}{ab}, \\ &= \pm \frac{(\sqrt{a} - \sqrt{b})^2}{ab}, \end{aligned}$$

$$\therefore\ x^{\frac{m-n}{mn}} = \frac{(\sqrt{a} - \sqrt{b})^4}{a^2 b^2},$$

$$\text{and } x = \left\{\frac{(\sqrt{a} - \sqrt{b})^4}{a^2 b^2}\right\}^{\frac{mn}{m-n}}.$$

4. From the first equation,

$$x^4 - 2x^2 y^2 + y^4 = 1 + 2xy + x^2 y^2,$$

$$\therefore\ x^2 - y^2 = \pm (xy + 1) \quad \ldots\ldots\ldots \quad (1).$$

Then, taking the positive sign, we have

$$\left.\begin{aligned} x^2 - 1 &= y^2 + xy, \\ \text{or, } (x-1)(x+1) &= y\,(x+y) \end{aligned}\right\} \quad \ldots\ldots \quad (2).$$

Again, from the second of the proposed equations,

$$(2y^2+1)(x+1) = x^3+y^3 = (x^2-xy+y^2)(x+y),$$

$\therefore$ dividing this by (2), we have

$$\frac{2y^2+1}{x-1} = \frac{x^2-xy+y^2}{y} \quad \ldots\ldots (3).$$

But from equations (2),

$$x^2-xy = 1+y^2,$$

$\therefore$ equation (3) becomes

$$\frac{2y^2+1}{x-1} = \frac{2y^2+1}{y} \quad \ldots\ldots\ldots (4).$$

Whence $2y^2+1=0$,

$$\therefore\ y = \pm\sqrt{-\tfrac{1}{2}}.$$

Substituting for y in (2), we get

$$x^2 \mp x\sqrt{-\tfrac{1}{2}} = \tfrac{1}{2}.$$

Whence,

$$\left.\begin{array}{l} x = \pm\tfrac{1}{2}\sqrt{\tfrac{3}{2}} \pm \tfrac{1}{2}\sqrt{-\tfrac{1}{2}} = \pm\tfrac{1}{2}\left(\sqrt{\tfrac{3}{2}}+\sqrt{-\tfrac{1}{2}}\right) \\ \text{and } y = \pm\sqrt{-\tfrac{1}{2}}, \end{array}\right\} \ldots (5).$$

which, taking upper signs together and lower signs together, give two pairs of simultaneous imaginary values of x and y.

Others may also be found.

5. Let x be the miles an hour he walked backwards;
then $4x$ are . . . . forwards.

Let, also, d be the given distance, and t the given time; then, by the question,

$$\frac{d}{4} \div x + \frac{3d}{4} \div 4x = t = \frac{d}{4} \div (x-\tfrac{1}{5}) + \frac{3d}{4} \div (4x+2),$$

$$\therefore \frac{1}{x} + \frac{3}{4x} = \frac{1}{x-\frac{1}{5}} + \frac{3}{4x+2},$$

$$\therefore \frac{7}{4x} = \frac{5}{5x-1} + \frac{3}{4x+2},$$

which being cleared of fractions gives $x=1$.

6. Let x£ be the debt,
y the price of a bond at the beginning of the year; then, since there must have been as many bonds taken as hun-

dreds in the loan, to produce the same interest, the value of those bonds was at first

$$\frac{x}{100} \times y,$$

and, by the question,

$$\frac{x}{100} \cdot \left(y - \frac{40\,y}{100}\right) = \frac{x\,y}{100} - 400 - 250,$$

$$\therefore \frac{2}{500}\,xy = 650,$$

$$xy = 325 \times 500 = 162500 \;.\;.\; (1).$$

Again, by the question,

$$\frac{x}{100} \times 50 = \frac{xy}{100} - 300 - 250,$$

$$\text{or } \frac{x}{2} = \frac{xy}{100} - 550. \;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \;(2).$$

But, by (1), $\frac{xy}{100} = 1625$,

$$\therefore \frac{x}{2} = 1625 - 550,$$

$$\therefore x = 2150l. \text{ the debt.}$$

Its interest $= \frac{1}{20} \times 2150 = 107l.\ 10s.$

Hence, $y = \dfrac{162500}{2150} = \dfrac{16250}{215} = \dfrac{3250}{43}$,

$\therefore$ the bonds were at $75\frac{25}{43}$ at the beginning of the year, and at the end, $75\frac{25}{43} - \frac{2}{5} \times 75\frac{25}{43} = \frac{3}{5} \times 75\frac{25}{43} = 45\frac{15}{43}$.

7. Let x be the number of prisoners at first,
and y shillings what he expected for his mill.

Hence, the expense of each man's provisions is

$a + (a + ar) + (a + ar + ar^2) +$ &c. to n terms $=$ A suppose.

Again, that of the fresh convicts is

$c\,(a + ar) + 2c\,(a + ar + ar^2) + \ldots$ to $(n-1)$ terms $=$ B suppose.

Now, the estimated expense was $n\,x \times a$,

the estimated labor was $n\,x \times p\,a$,

$\therefore\ n\,a\,(p-1)\,x =$ expected gain.

Hence, by the question,

$$n x \times \overline{pa + c} \times \overline{pa+2c} \times \overline{pa+} \ldots (n-1) \text{ terms} - x\,A - B = n\,a\,(p-1)\,x,$$

$$\therefore x = \frac{B - p\,a\,c\{1+2+\ldots(n-1) \text{ terms}\}}{n\,a - A} \ldots (1).$$

$$\text{But}A = na + ar + (ar+ar^2) + ar + ar^2 + ar^3 + .. ar + ar^2 + .. ar^{n-1}$$

$$\therefore \frac{A - na}{r} = a + (a+ar) + (a+ar+ar^2) + \ldots(a+ar+\ldots ar^{n-2})$$

$$= A - (a + ar + \ldots ar^{n-1}),$$

$$= A - a\,\frac{r^n - 1}{r-1},$$

$$\therefore A - na = r\,A - ar.\,\frac{r^n - 1}{r-1},$$

$$\therefore A = \frac{na - ar.\,\dfrac{r^n-1}{r-1}}{1-r},$$

$$\therefore na - A = \frac{ar}{1-r}.\left(\frac{1-r^n}{1-r} - n\right) \ldots \ldots \ldots (2).$$

$$\text{Also, } \frac{B}{ac} = (1+r) + (1+r+r^2) + \ldots.(1+r+r^2+\ldots.r^{n-1}),$$

$$= n-1 + r\{1 + (1+r) + (1+r+r^2) + ..(1+r+\ldots r^{n-2})\},$$

$$= n - 1 + r + r\left\{\frac{B}{ac} - (1+r+\ldots..r^{n-1})\right\},$$

$$= n-1 + r + r\left\{\frac{B}{ac} - \frac{r^n - 1}{r-1}\right\},$$

$$\therefore \frac{B}{ac} = \frac{n-1+r-r.\,\dfrac{r^n-1}{r-1}}{1-r}$$

$$\therefore B = ac \times \frac{n-1-r^2.\,\dfrac{1-r^{n-1}}{1-r}}{1-r} \ldots \ldots (3).$$

$$\text{Also, } 1+2+\ldots(n-1) \text{ terms} = \frac{n^2}{2}.$$

Consequently, substituting in (1), we finally get

$$x = \frac{c}{r} \times \frac{n-1-r^2.\,\dfrac{1-r^{n-1}}{1-r} - \dfrac{pn^2}{2}.\,(1-r)}{\dfrac{1-r^n}{1-r} - n}.$$

ST. JOHN'S COLLEGE, 1824.

1. Multiply by 21, and reduce. Then multiply by 100.

Ans. $x = 8$.

2. Make $\frac{1}{x} = u, \frac{1}{y} = v, \frac{1}{z} = w,$

then the equations are

$$\left.\begin{array}{l} 3u - \frac{4}{5}v + w = \frac{38}{5} \\ \frac{1}{3}u + \frac{1}{2}v + 2w = \frac{61}{6} \\ \frac{4}{5}u - \frac{1}{2}v + 4w = \frac{161}{10} \end{array}\right\}.$$

From these it is easily found, that

$$u = 2,\ v = 3,\ w = 4,$$

$$\therefore\ x = \tfrac{1}{2},\ y = \tfrac{1}{3},\ z = \tfrac{1}{4}.$$

3. First multiply by 50; then

$$\frac{1350 - 75\sqrt{x} - 50x}{5\sqrt{x} + 8} = 1 + 4.\ \frac{148\sqrt{x} - 10x\sqrt{x}}{4x - 7},$$

$$\therefore \frac{1342 - 80\sqrt{x} - 50x}{5\sqrt{x} + 8} = \frac{592\sqrt{x} - 40x\sqrt{x}}{4x - 7},$$

$$\frac{671 - 40\sqrt{x} - 25x}{5\sqrt{x} + 8} = \frac{296\sqrt{x} - 20x\sqrt{x}}{4x - 7}.$$

This gives $x - \frac{2088}{1379}\sqrt{x} = \frac{4697}{1379}$.

Solving this, $\sqrt{x} = \frac{1044 \pm \sqrt{7567099}}{1379}$,

$$\therefore x = \left(\frac{1044 \pm \sqrt{7567099}}{1379}\right)^2,$$

which is easily reduced to decimals.

4. From the first,

$$\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 - x\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right) = \frac{y}{2} - x\sqrt{\frac{y}{2}},$$

$\therefore$ completing the square, and extracting the root,

$$\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}-\frac{x}{2}=\pm\sqrt{\left(\frac{y}{2}-x\sqrt{\frac{y}{2}}+\frac{x^2}{4}\right)}$$

$$=\pm\left(\sqrt{\frac{y}{2}}-\frac{x}{2}\right),$$

$$\left.\begin{array}{l}\therefore \sqrt{\dfrac{x}{y}}-\sqrt{\dfrac{y}{x}}=\sqrt{\dfrac{y}{2}} \\ \text{and } \sqrt{\dfrac{x}{y}}-\sqrt{\dfrac{y}{x}}=x-\sqrt{\dfrac{y}{2}}\end{array}\right\}\begin{array}{l}\ldots\ldots(1) \\ \ldots\ldots(2).\end{array}$$

Each of these equations is *simultaneous* with the second of the proposed equations, viz.

$$9\sqrt{\frac{x}{y}}+3\sqrt{\frac{y}{x}}=(21\sqrt{2x}-1)\times\frac{1}{2}\sqrt{\frac{y}{x}}+\frac{1}{2\sqrt{(xy)}}\ (3)$$

Multiplying (1) and (3) by $\sqrt{(xy)}$, we get

$$\left.\begin{array}{l}x-y=\dfrac{y\sqrt{x}}{\sqrt{2}} \\ \text{and } 9x+3y=(21\sqrt{2x}-1).\dfrac{y}{2}+\dfrac{1}{2}\end{array}\right\},$$

$$\therefore y=\frac{x\sqrt{2}}{\sqrt{2}+\sqrt{x}}.$$

Also, $18x+6y=21y\sqrt{2x}-y+1$,

$$\therefore (7y-21\sqrt{2x})\,y=1-18x,$$

$$\therefore \frac{x\sqrt{2}}{\sqrt{2}+\sqrt{x}}.\,(6-21\sqrt{2x})=1-18x,$$

$$\therefore 7x\sqrt{2}-42x\sqrt{x}=\sqrt{2}-18\sqrt{2}.\,x=\sqrt{x}-18x\sqrt{x},$$

$$25x\sqrt{2}-24x\sqrt{x}=\sqrt{x}+\sqrt{2},$$

$$\therefore \sqrt{x}\,(24x-\sqrt{2x}-1)=24x\sqrt{2}-2\sqrt{x}-\sqrt{2},$$

$$=\sqrt{2}\,(24x-\sqrt{2x}-1),$$

$$\therefore (\sqrt{x}-\sqrt{2})\,(24x-\sqrt{2x}-1)=0,$$

$$\therefore x=2,$$

$$\text{and } 24x-\sqrt{2x}-1=0,$$

$$\therefore x-\frac{1}{12\sqrt{2}}.\sqrt{x}=\frac{1}{24},$$

$\therefore$ solving the quadratic,

$$\sqrt{x} = \frac{1}{24\sqrt{2}} \pm \sqrt{\left(\frac{1}{24^2 \times 2} + \frac{1}{24}\right)},$$

$$= \frac{1 \pm 7}{24\sqrt{2}} = \frac{1}{3\sqrt{2}} \text{ and } -\frac{1}{4\sqrt{2}},$$

$$\therefore x = \frac{1}{18} \text{ and } \frac{1}{32}.$$

Hence,

when $x = 2$, $y = \dfrac{2\sqrt{2}}{\sqrt{2}+\sqrt{2}} = 1$,

when $x = \dfrac{1}{18}$, $y = \dfrac{1}{9\sqrt{2}} \div \left(\sqrt{2} + \dfrac{1}{3\sqrt{2}}\right) = \dfrac{1}{9\sqrt{2}} \times \dfrac{3\sqrt{2}}{7} = \dfrac{1}{21}$,

when $x = \dfrac{1}{32}$, $y = \dfrac{1}{16\sqrt{2}} \div \left(\sqrt{2} + \dfrac{1}{4\sqrt{2}}\right) = \dfrac{1}{16\sqrt{2}} \times \dfrac{4\sqrt{2}}{9} = \dfrac{1}{36}$,

which are three pairs of simultaneous values of x and y.

Again, take also the simultaneous equations

$$\left.\begin{aligned} &\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} = x - \sqrt{\frac{y}{2}} \\ \text{and } 9&\sqrt{\frac{x}{y}} + 3\sqrt{\frac{y}{x}} = (21\sqrt{2}\,x - 1)\frac{1}{2}\sqrt{\frac{y}{x}} + \frac{1}{2\sqrt{(xy)}} \end{aligned}\right\}.$$

Multiplying by $\sqrt{(xy)}$, we get

$$\left.\begin{aligned} &x - y = x\sqrt{(xy)} - y\sqrt{\frac{x}{2}} \\ &\text{and } (7 - 21\sqrt{2}\,x)\,y = 1 - 18\,x \end{aligned}\right\} \begin{aligned} &\cdots\cdots (4) \\ &\cdots\cdots (5) \end{aligned}$$

$$\therefore\ 7\,(1 - 3\sqrt{2}\,x)\,y = (1 - 3\sqrt{2}\,x)\,(1 + 3\sqrt{2}\,x),$$

$$\therefore\ 1 - 3\sqrt{2}\,x = 0, \text{ or } x = \frac{1}{18}.$$

Substituting this value for x in equation (4), we have

$$\frac{1}{18} - y = \frac{1}{18}\sqrt{\left(\frac{1}{18} \cdot y\right)} - \frac{y}{6},$$

$$1 - 18\,y = \frac{1}{3}\sqrt{\frac{y}{2}} - 3\,y,$$

$$\therefore\ 15\,y + \frac{1}{3\sqrt{2}} \cdot \sqrt{y} = 1,$$

$$\therefore\ y + \frac{1}{45\sqrt{2}} \cdot \sqrt{y} = \frac{1}{15},$$

$$y + \frac{1}{45\sqrt{2}}.\sqrt{y} + \left(\frac{1}{90\sqrt{2}}\right)^2 = \frac{1}{15} + \frac{1}{90^2 \times 2},$$

$$= \frac{1081}{90^2 \times 2},$$

$$\therefore \sqrt{y} = \frac{-1 \pm \sqrt{1081}}{90\sqrt{2}},$$

$$\therefore y = \frac{(1 \mp \sqrt{1081})^2}{16200}.$$

From the equations

$$\left.\begin{aligned} 7y &= 1 + 3\sqrt{2}.\,x \\ \sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} &= x - \sqrt{\frac{y}{x}} \end{aligned}\right\},$$

other simultaneous values of x and y may be found, very nearly as above.

5. Let x be the number of bags intended to be purchased.

Then, $\frac{x}{5} \times 8\,£ =$ the sum transmitted.

and, by the question,

$\frac{8x}{5} \div (x-18) =$ cost of each bag.

Also,

$$\left(\frac{1}{3}x - 5\frac{1}{4}\right) \times \frac{8}{5} \times \frac{x}{x-18} = \left(\frac{1}{3}x - 5\frac{1}{4}\right)\frac{8}{5} + 10\frac{7}{20},$$

$$\left(\frac{4}{3}x - 21\right)\frac{8x}{x-18} = \left(\frac{4}{3}x - 21\right)8 + 207,$$

$$(4x - 63)\frac{8x}{x-18} = (4x-63)\,8 + 621,$$

$$\therefore (4x-63)\,8 \times 18 = 621\,x - 18 \times 621,$$

$$\therefore (4x-63)\,16 = 69\,x - 1242,$$

$$5x = -63 \times 16 + 1242 = 234,$$

$$\therefore x = 46\tfrac{4}{5} \text{ bags}.$$

6. Let x be miles an hour at which A travels,

y . . . . B . . . .

and z the miles C is distant from D.

Then, the time B travels, before they meet the first time, is

$$\frac{20}{y} \text{ hours};$$

the time of A reaching the place of rencontre, is

$$\frac{z-20}{x};$$

and, by the question,

$$\frac{z-20}{x} = \frac{20}{y} + 3 \ldots\ldots\ldots (1).$$

Again, A reaches D in $\frac{z}{x}$ hours, and B reaches C in $\frac{z}{y}$ hours;

$\therefore$ by the question,

$$\frac{z}{x} = \frac{z}{y} + 3 - 1 = \frac{z}{y} + 2 \ldots\ldots\ldots (2).$$

Lastly, whilst A was performing $\frac{6}{7}z$ miles, B had time to go

$$\frac{z}{7} + 28 + 3y \text{ miles};$$

$\therefore$ since distance $\propto$ rate,

$$\tfrac{6}{7}z : \tfrac{1}{7}z + 28 + 3y :: x : y,$$

$$\therefore \tfrac{6}{7}yz = \tfrac{1}{7}xz + 28x + 3xy \ldots\ldots (3).$$

The equations (1), (2), (3), are reducible to

$$\left.\begin{array}{l} yz - 20y = 20x + 3xy \\ yz - xz = 2xy \\ 6yz - xz = 196x + 21xy \end{array}\right\} \begin{array}{l} \ldots\ldots (4). \\ \ldots\ldots (5). \\ \ldots\ldots (6). \end{array}$$

From (4) $z = 20\frac{x}{y} + 3x + 20$; substituting this in (5) and (6),

$$\left.\begin{array}{r} (y-x)\left(20\frac{x}{y} + 3x + 20\right) = 2xy \\ \text{and } (6y-x)\left(20\frac{x}{y} + 3x + 20\right) = 196x + 21xy \end{array}\right\} \begin{array}{l} (7). \\ (8). \end{array}$$

Multiplying out, these become, when reduced,

$$\left.\begin{array}{r} xy + 20y - 20\frac{x^2}{y} - 3x^2 = 0 \\ \text{and } 3xy - 120y + 20\frac{x^2}{y} + 3x^2 + 96x = 0 \end{array}\right\} \begin{array}{l} (9). \\ (10). \end{array}$$

Adding (9) and (10) together, we get

$$4\,xy - 100\,y + 96\,x = 0,$$

$$\text{or } xy - 25\,y + 24\,x = 0,$$

$$\therefore x = \frac{25y}{24+y}. \quad \ldots\ldots\ldots\ldots \quad (11)$$

which, being substituted in (9), gives

$$\frac{25\,y^2}{24+y} + 20y - \frac{20 \times 25^2\,y^2}{y\,(24+y)^2} - \frac{3 \times 25^2\,y^2}{(24+y)^2} = 0,$$

$$\therefore \frac{5y}{24+y} + 4 - \frac{4 \times 25^2}{(24+y)^2} - \frac{3 \times 125\,y}{(24+y)^2} = 0,$$

$$\therefore (9\,y + 96)\,(24+y) = 2500 + 375\,y,$$

$$\therefore 216\,y + 2304 + 9\,y^2 + 96\,y = 2500 + 375\,y,$$

$$9\,y^2 - 63\,y = \left.\begin{matrix}2500\\2304\end{matrix}\right\} = 196,$$

$$\therefore y^2 - 7\,y = \frac{196}{9},$$

$$\therefore y = \frac{7}{2} \pm \sqrt{\left(\frac{49}{4} + \frac{196}{9}\right)} = \frac{7}{2} \pm \frac{36}{6} = \frac{28}{3} = 9\,\frac{1}{3},$$

Hence, by equation (11),

$$x = 25 \times \frac{28}{3} \div 24 + \frac{28}{3} = \frac{25 \times 28}{100} = 7.$$

$$\text{Also, } z = 20 \times \frac{x}{y} + 3\,x + 20,$$

$$= 20 \times \tfrac{3}{4} + 21 + 20 = 56 \text{ miles.}$$

7. Let x be the pounds the first creditor can claim,
y the common difference,
and z the number of creditors;
then the whole debt is (*Wood*, art. 212),

$$\{2\,x + (z-1)\,y\}\frac{z}{2}l.$$

for which the effects being y shillings in the pound, we have

$$\text{effects} = \{2\,x + (z-1)\,y\}\,\frac{yz}{40}.$$

Again, the last creditor would receive $\{x+(z-1)y\}\frac{y}{20}l$.

Hence, by the question,

$$\{x+(z-1)y\}\frac{y}{20} = \tfrac{1}{2}(x+x+y) = x+\frac{y}{2} \quad . \; . \; . \; (1).$$

Again, the last creditor having failed of his claim,

$$\{2x+(z-2)y\}\frac{z-1}{2}$$

is now the total debt; $\because$ debt $\propto \frac{1}{\text{dividend}}$,

$$\therefore\ \{2x+(z-1)y\}\frac{z}{2} : \{2x+(z-2)y\}\frac{z-1}{2} :: \frac{1}{y} : \frac{1}{y+\frac{8}{3}},$$

or, $2xz+z(z-1)y : 2xz-2x+(z-1)(z-z)y :: 3y+8 : 3y$,

Hence,

$$3(z^2-z)y^2+6xy = 3(z^2-3x+2)y^2+16xz-16x$$
$$+8(z^2-3z+2)y,$$

$$\therefore\ (6z-6)y^2+6xy-16xz+16x-8(z^2-3z+2)y = 0,$$

or, $3(z-1)y^2+3xy- 8xz+ 8x-4(z^2-3z+2)y = 0.(2)$:

Lastly,

$$\{x+(z-1)y\}\frac{y}{20} : (x+y)\frac{y+\frac{8}{3}}{20} :: 45 : 28,$$

$$\therefore\ 28y\{x+(z-1)y\} = 15(x+y)(3y+8) \quad . \; . \; . \; . \; . \; . \; (3).$$

Divide (3) by (1), and the result is

$$560 = \frac{30(x+y(3y+8)}{2x+y}$$

$$\therefore\ 112x+56y = 9xy+24x+9y^2+24y,$$

$$\therefore\ 9y^2-32y+9xy-88x = 0,$$

and hence,

$$x = \frac{9y^2-32y}{88-9y} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (4).$$

Again, eliminate z from (1) and (2), thence obtaining an equation in x and y. Substitute in this the value of x from (4), and the resulting equation in y will give $y = 8$. Whence by (4) it will be found that $x = 20$; and then by (3) we get $z = 6$.

ST. JOHN'S COLLEGE, 1825.

1. Multiply by 36; then

$$8x+34-\frac{468x-72}{17x-32}+12x=21x-x-16,$$

whence $x=4$.

2. From the first $(x+y)^{\frac{1}{3}}.\dfrac{x+y}{xy}=\dfrac{8}{63}$;

from the second $(x-y)^{\frac{1}{3}}.\dfrac{x-y}{xy}=\dfrac{4}{63}$.

$\therefore$ by division,

$$\left(\frac{x+y}{x-y}\right)^{\frac{4}{3}}=2,$$

$$\therefore\ x+y=8^{\frac{1}{4}}.\ (x-y),$$

$$\text{whence } x=\frac{8^{\frac{1}{4}}+1}{8^{\frac{1}{4}}-1}.\ y.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (1)$$

and substituting in either of the proposed equations for x, the resulting equation in y will give y; and x will thence be obtained by means of equation (1).

Another Method.

Since the equations are homogeneous, make $y=ux$; then the equations become

$$\left.\begin{aligned}\frac{(x+ux)^{\frac{4}{3}}}{x^2u}&=\frac{8}{63}\\ \text{and}\quad \frac{(x-ux)^{\frac{4}{3}}}{xu}&=\frac{4}{63}\end{aligned}\right\}\ \therefore\ \left\{\begin{aligned}\frac{(1+u)^{\frac{4}{3}}}{ux^{\frac{2}{3}}}&=\frac{8}{63}\\ \frac{(1-u)^{\frac{4}{3}}}{ux^{\frac{2}{3}}}&=\frac{4}{63}\end{aligned}\right\}$$

&c. as is obvious.

3. Divide by $x^{\frac{p+q}{2pq}}$, and there results

$$1-\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\left(x^{\frac{q-p}{2pq}}+x^{\frac{p-q}{2pq}}\right)=0.$$

Let $\dfrac{p-q}{2pq}=m$; then

$$x^m+x^{-m}=2.\ \frac{a^2+b^2}{a^2-b^2},$$

$$\therefore\ x^{2m}-2\,\frac{a^2+b^2}{a^2-b^2}\,x^m=-1,$$

$\therefore$ solving the quadratic,

$$x^m=\frac{a^2+b^2}{a^2-b^2}\pm\sqrt{\frac{(a^2+b^2)^2-(a^2-b^2)^2}{(a^2-b^2)^2}},$$

$$=\frac{a^2+b^2\pm 2ab}{a^2-b^2}=\frac{(a\pm b)^2}{a^2-b^2},$$

$$=\frac{a+b}{a-b}\text{ and }\frac{a-b}{a+b},$$

$$\therefore\ x=\left(\frac{a+b}{a-b}\right)^{\frac{2pq}{p-q}}\text{ and }\left(\frac{a-b}{a+b}\right)^{\frac{2pq}{p-q}},$$

x also $=0$.

4. From the first,

$$3x+y-2=x\sqrt{\left(\frac{5x^2}{4}-2y+8\right)}.$$

Squaring and reducing, we get

$$y^2+(2x^2+6x-4)\,y=\frac{5x^4}{4}-x^2+12x-4.$$

Completing the square of the left-hand member we get, after deduction of the right,

$$y^2+(2x^2+6x-4)\,y+(x^2+3x-2)^2=\frac{9}{4}x^4+6x^3+4x^2.$$

Extracting the root

$$y+x^2+3x-2=\pm\left(\frac{3}{2}x^2+2x\right),$$

$$\left.\begin{aligned}\therefore\ y&=\tfrac{1}{2}x^2-x+2\\ \text{and } y&=-\tfrac{5}{2}x^2-5x+2\end{aligned}\right\}\begin{aligned}&\ldots\ldots(1).\\&\ldots\ldots(2).\end{aligned}$$

X

Again, from the second of the proposed equations, we have

$$\frac{x+y}{2x} - 2x + 3 = \left(\frac{3}{4}x - \frac{3y}{2x}\right)\sqrt{(2+y)},$$

$$\therefore x + y - 4x^2 + 6x = (\tfrac{3}{2}x^2 - 3y)\sqrt{(x+y)},$$

$$\therefore x + y - (\tfrac{3}{2}x^2 - 3y)\sqrt{(x+y)} = 4x^2 - 6x,$$

$$\text{But } \tfrac{3}{2}x^2 - 3\sqrt{y} = \tfrac{3}{2}x^2 - \tfrac{3}{2}x^2 + 3x - 6,$$

$$= 3x - 6,$$

$$\therefore x + y - (3x + 6)\sqrt{(x+y)} = 4x^2 - 6x,$$

$$\therefore \sqrt{(x+y)} = \tfrac{3}{2}x - 3 \pm \sqrt{(\tfrac{25}{4}x^2 - 19x + 9)},$$

$$= \tfrac{3}{2}x - 3 \pm (\tfrac{5}{2}x - 3),$$

$$= 4x - 6 \text{ and } -x,$$

$$\therefore x + y = 16x^2 - 48x + 36 \text{ and } x^2,$$

$$\left.\begin{array}{l}\therefore y = 16x^2 - 49x + 36 \\ \text{also } y = x^2 - x\end{array}\right\} \begin{array}{l}\ldots\ldots (3). \\ \ldots\ldots (4).\end{array}$$

From (1) and (3),

$$\tfrac{31}{2}x^2 - 48x + 34 = 0,$$

$$\therefore x = \frac{48}{31} \pm \sqrt{\left(\frac{48^2}{31^2} - \frac{68}{31}\right)},$$

$$= \frac{48 \pm 14}{31} = 2 \text{ and } \frac{34}{31},$$

$$\text{Hence } y = \tfrac{1}{2}x^2 - x + 2,$$

$$= 2 \text{ and } \tfrac{1446}{961},$$

$\therefore$ two pairs of simultaneous values of x and y are

$$\left.\begin{array}{l}x = 2 \\ y = 2\end{array}\right\} \text{ and } \left.\begin{array}{l}x = \dfrac{34}{31} \\ y = \dfrac{1446}{961}\end{array}\right\} \ldots\ldots (5).$$

Again, from (1) and (4), we get

$$x^2 - x = \tfrac{1}{2}x^2 - x + 2,$$

$$\therefore x^2 = 4,$$

$$\text{and } x = \pm 2,$$

$$\therefore y = 4 - 2 = 2 \text{ and } = 4 + 2 = 6,$$

$\therefore$ two more pairs of simultaneous values are

$$\left.\begin{array}{l}x = 2 \\ y = 2\end{array}\right\} \left.\begin{array}{l}x = -2 \\ y = 6\end{array}\right\} \ldots\ldots (6).$$

Again, from equations (2) and (3),

$$-\tfrac{5}{2}x^2 - 5x + 2 = 16x^2 - 49x + 36,$$

$$\tfrac{27}{2}x^2 - 44x = -34,$$

$$\therefore x^2 - \tfrac{88}{27}x = -\tfrac{68}{27},$$

$$\therefore x = \frac{44}{27} \pm \sqrt{\frac{44^2 - 68 \times 27}{27^2}},$$

$$= \frac{44 \pm 10}{27} = 2 \text{ and } \frac{34}{27},$$

$$\therefore y = -\tfrac{5}{2}x^2 - 5x + 2,$$

$$= -10 - 10 + 2 = -18,$$

$$\text{and} = -\frac{5}{2}\,\frac{34^2}{27^2} - 5.\frac{34}{27} + 2 = -\frac{5022}{27} = -\frac{558}{3} = -186.$$

Whence two more pairs of simultaneous values of x and y, are

$$\left.\begin{array}{l} x = 2 \\ y = -18 \end{array}\right\} \text{and } \left.\begin{array}{l} x = \dfrac{34}{27} \\ y = -186 \end{array}\right\} \ldots\ldots (7).$$

Two other sets of simultaneous values may be found, in like manner, from (2) and (4). See *Hind's* Algebra, Second Edition, p. 502, 56.

5. Let x£. be his annual expenditure,

y£. be the yearly value of his produce; then, by the equation,

$$y - x - \frac{x}{6} = 30 \ldots\ldots (1).$$

In the next year, the rent $= 50 - \frac{1}{5} \times 50 = 40$,

$$\text{taxes} = \frac{x}{12},$$

$$\text{produce} = y + \frac{y}{3} = \frac{4y}{3},$$

$\therefore$ by the question,

$$\frac{4y}{3} - x - \frac{x}{12} = 40 + 20 + 5 = 65 \ldots\ldots (2).$$

These first become

$$\left.\begin{array}{l} 6y - 7x = 180 \\ 16y - 13x = 780 \end{array}\right\} \begin{array}{l} \ldots\ldots (4). \\ \ldots\ldots (5). \end{array}$$

Multiply first by 8, and second by 3, and subtract; then

$$17x = 900,$$

$$\therefore x = \tfrac{900}{17}l. = 52\tfrac{16}{17}l.,$$

$$\therefore y = \tfrac{1050}{17} + 30 = 91\tfrac{13}{17}l.,$$

6. Let B ride x miles an hour,

C . . . y . . . .

and $2z$ be the miles distant from London to Newmarket.

Hence, if t be the hours before A and B meet,

$$10t + xt = z, \ldots \ldots (1).$$

also C overtakes B in $t + 4$ hours,

$$\therefore y(t+4) = z + x(t+4) \ldots \ldots (2).$$

Again, if t' be the hours before A and B meet, on their return,

$$10t' + xt' = 5z \ldots \ldots (3).$$

$$\text{and } 10(t'+1) + y(t'+1) = 6z \ldots \ldots (4).$$

$$\text{Lastly, } 10(t+t') = 2z + z + xt \ldots \ldots (5).$$

From these five equations, x, y, z, t, and t', may be found as follows:

From the (1) and (3),

$$t = \frac{z}{10+x}, \quad t' = \frac{5z}{10+x},$$

$\therefore$ by substitution in (2), (4), (5), and reduction,

$$\left.\begin{array}{l} 4x^2 + 40x - 4xy + 2xz - 40y - zy + 10z = 0 \\ 10x + xy - 6xz + 10y + 5yz - 10z + 100 = 0 \\ \quad \text{and } 2x = 15, \end{array}\right\} (6).$$

$\therefore x = 7\frac{1}{2}$ miles an hour,

and substituting the values of y, z, t, t' may easily be found.

7. Let $4x$ = quantity of fluid in P, at first; then

$21x$ = ... Q . . . .

Also, let $4y$ be what A pumped out of P; then

$$\left.\begin{array}{l} 4x - 4y = \text{what was left in P}, \\ 21x + 4y = \text{quantity now in Q} \end{array}\right\}.$$

Also, after B had pumped,

$$\left.\begin{array}{r} x - y = \text{what was left in P}, \\ 21x + 4y + 3x - 3y = 24x + y = \text{what was then in Q} \end{array}\right\} \ldots (a).$$

Again, let $u = \dfrac{\text{Q}^y \text{ of wine}}{\text{Q}^y \text{ of spirits}}$ in P, at first,

v = ditto in Q, at first,

$$\text{then } \because u+1 = \frac{\text{Q. of wine} + \text{Q. of spirits}}{\text{Q. of spirits}} \text{ in P} = \frac{4\,x}{\text{Q. of spirits}},$$

$$\text{and } v+1 = \quad \text{ditto} \quad \text{in Q} = \frac{21x}{\text{Q. of spirits}},$$

$$\left.\begin{aligned} \therefore \text{ Q of spirits in P, at first} &= \frac{4x}{u+1} \\ \text{ditto} \quad \text{in Q,} \quad &= \frac{21x}{v+1} \end{aligned}\right\} \ldots\ldots (b),$$

$$\left.\begin{aligned} \text{and hence Q of wine in P, at first} &= \frac{4\,ux}{u+1} \\ \text{ditto} \quad \text{in Q,} \quad &= \frac{21\,vx}{v+1} \end{aligned}\right\} \ldots\ldots (c).$$

Hence, also, after B had pumped,

$$\left.\begin{aligned} \text{spirits in Q} &= \frac{21\,x}{v+1} + \frac{3\,x+y}{u+1} \\ \text{and wine in Q} &= \frac{21vx}{v+1} + \frac{(3\,x+y)\,u}{u+1} \end{aligned}\right\} \ldots\ldots (d),$$

the parts added by pumping being found by Rule of Three statements.

Again, if s denote the strength of wine, then, by the question, $3\,s$ is that of spirits; and then, also, by the question,

$$\left(\frac{21\,x}{v+1} + \frac{3\,x+y}{u+1}\right) 3\,s + \left\{\frac{21\,vx}{v+1} + \frac{(3\,x+y)\,u}{u+1}\right\} s = \frac{12}{13}$$
$$\left(\frac{21\,x}{v+1} + 3\,s + \frac{21\,vx}{v+1} \times s\right),$$

which, when reduced, gives

$$\frac{21\,x}{13} \cdot \frac{v+3}{v+1} + (3\,x+y) \cdot \frac{u+3}{u+1} = 0 \ldots (1).$$

Again, supposing B had pumped $3x-3y$ fluid from Q into P, after A had pumped $4\,y$ into Q, then

$$\text{spirits in P} = \frac{4\,x}{u+1} - \frac{4\,y}{u+1} + \frac{3\,x-3\,y}{21\,x+4\,y}\left(\frac{21\,x}{v+1} + \frac{4\,y}{u+1}\right)$$
$$= (x-y)\left\{\frac{4}{u+1} + \frac{3}{21x+4y}\left(\frac{21\,x}{v+1} + \frac{4\,y}{u+1}\right)\right\},$$
$$\therefore \text{ wine in P} = 4\,x - 4\,y + 3\,x - 3\,y - \text{spirits in P}$$
$$= (x-y)\left\{7 - \frac{4}{u+1} - \frac{3}{21x+4y}\left(\frac{21\,x}{v+1} + \frac{4\,y}{u+1}\right)\right\}$$

$$= (x-y)\left\{\frac{7u+3}{u+1} - \frac{3}{21x+4y}\left(\frac{21x}{v+1}+\frac{4y}{u+1}\right)\right\},$$

Hence, by the question,

$$3s(x-y)\left\{\frac{4}{u+1}+\frac{3}{21x+4y}\left(\frac{21x}{v+1}+\frac{4y}{u+1}\right)\right\}+s(x-y)$$
$$\left\{\frac{7u+3}{u+1}-\frac{3}{21x+4y}\left(\frac{21x}{v+1}+\frac{4y}{u+1}\right)\right\}=\sqrt{\left\{\left(3s.\ \frac{4x}{u+1}\right.\right.}$$
$$\overline{\left.\left.+\,s.\ \frac{4ux}{u+1}\right).\left(3s.\ \frac{21x}{v+1}+s.\ \frac{21vx}{v+1}\right)\right\}},$$

$$\text{or, } (x-y)\left\{\frac{12}{u+1}+\frac{9}{21x+4y}\left(\frac{21x}{v+1}+\frac{4y}{u+1}\right)+\frac{7u+3}{u+1}-\right.$$
$$\left.\frac{3}{21x+4y}\left(\frac{21x}{v+1}+\frac{4y}{u+1}\right)\right\}=\sqrt{\left(\frac{12x+4ux}{u+1}\cdot\frac{63x+21vx}{v+1}\right)},$$

$$\text{or, } (x-y)\left\{\frac{7u+15}{u+1}+\frac{6}{21x+4y}\left(\frac{21x}{v+1}+\frac{4y}{u+1}\right)\right\}=$$
$$2x\sqrt{\left(\frac{u+3}{u+1}\cdot\frac{v+3}{v+1}\right)}.\ \ldots\ldots\ldots\ (2).$$

Lastly, B would have pumped out the same quantity of wine as he did before of spirits. But after A had pumped,

$$\text{wine in Q} = \frac{21vx}{v+1}+\frac{4uy}{u+1},$$

$$\therefore \text{ wine pumped by Q} = \frac{3x-3y}{21x+4y}\cdot\left(\frac{21vx}{v+1}+\frac{4uy}{u+1}\right),$$

Also, spirits pumped by $Q = \dfrac{3x-3y}{u+1}$,

$$\therefore \frac{21vx}{v+1}+\frac{4uy}{u+1} = \frac{21x+4y}{u+1},$$

$$\text{or, } \frac{21vx}{v+1} = \frac{21x+4y-4uy}{u+1}\ \ldots\ (3).$$

Although there are but three equations for the four unknowns x, y, u, v, these may still be determined. For $\because$ (1), (2), (3) are homogeneous with respect to x, y; making

$$y = wx,$$

we obtain

$$\frac{21}{13}\cdot\frac{v+3}{v+1}+(3+w).\ \frac{u+3}{u+1}=0.\ \ldots\ (4).$$

$$\left.\begin{aligned}(1-w)\left\{\frac{7u+15}{u+1}+\frac{6}{21+4w}\left(\frac{21}{v+1}+\frac{4w}{u+1}\right)\right\}\\=2\sqrt{\left(21.\frac{u+3}{u+1}.\frac{v+3}{v+1}\right)}\ldots\ldots\\ \frac{21v}{v+1}=\frac{21+4w-4uw}{u+1}\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\begin{aligned}&\ldots(5).\\&\ldots(6).\end{aligned}$$

from these u, v, w being found, the quantities required may then be obtained; it being observed that (6) gives

$$w=\frac{21}{4}.\frac{uv-1}{(1-u)(1+v)}.$$

ST. JOHN'S COLLEGE, 1826.

1. First multiply by $x+1$, then reduce, then multiply by $3x+2$, and we have $x=\frac{27}{8}$.

2. Ans. $x=3$, $y=2$.

3. Since $a^{4b}x-2a^{4b}\sqrt{x}+a^{4b}+x-2\sqrt{x}+1=2x+2$,

$$\therefore\ x(1-a^{4b})+2(1+a^{4b})\sqrt{x}=-(1-a^{4b}),$$

$$\therefore\ x+2.\frac{1+a^{4b}}{1-a^{4b}}\sqrt{x}=-1,$$

$\therefore$ completing the square, &c.

$$\sqrt{x}+\frac{1+a^{4b}}{1-a^{4b}}=\pm\sqrt{\frac{(1+a^{4b})^2-(1-a^{4b})^2}{(1-a^{4b})^2}},$$

$$=\pm\frac{2a^{2b}}{1-a^{4b}},$$

$$\therefore\ \sqrt{x}=-\frac{1-2a^{2b}+a^{4b}}{1-a^{4b}}\text{ and }-\frac{1+2a^{2b}+a^{4b}}{1-a^{4b}},$$

$$=-\frac{(1-a^{2b})^2}{1-a^{4b}}\text{ and }-\frac{(1+a^{2b})^2}{1-a^{4b}},$$

$$=-\frac{1-a^{2b}}{1+a^{2b}}\text{ and }-\frac{1+a^{2b}}{1-a^{2b}},$$

$$\therefore\ x=\left(\frac{1-a^{2b}}{1+a^{2b}}\right)^2\text{ and }\left(\frac{1+a^{2b}}{1-a^{2b}}\right)^2.$$

4. First divide the second by the first, and we get

$$\frac{9}{2x+\sqrt{(x^2-y^2)}}=\frac{y}{14}\left\{\frac{x+y}{2}-\tfrac{1}{2}\sqrt{(x^2-y^2)}+\frac{x-y}{2}\right\},$$

(Private Tutor,) p. 98.

$$\therefore \frac{2x-\sqrt{(x^2-y^2)}}{3x^2+y^2}=\frac{y}{252}\left\{2x-\sqrt{(x^2-y^2)}\right\},$$

$$\left.\begin{aligned}\therefore\ 2x &= \sqrt{(x^2-y^2)}\\ \text{and } y^3+3x^2y &= 252\end{aligned}\right\}\ldots\ldots(1).$$

each of which must be combined with either of the given equations.

The former gives $y^2=-3x^2$, which, combined with the second of the given equations, leads to an absurdity. But squaring the second of the given equations, we get

$$\left(\frac{x+y}{2}\right)^3+\left(\frac{x-y}{2}\right)^3+2\left(\frac{x^2-y^2}{4}\right)^{\frac{3}{2}}=81,$$

$$\therefore\ (x+y)^3+(x-y)^3+2(x^2-y^2)^{\frac{3}{2}}=648,$$

$$\therefore\ x^3+3xy^2+(x^2-y^2)^{\frac{3}{2}}=324\ \ldots\ldots(2).$$

Adding (1) to, and subtracting it from (2), we get

$$\left.\begin{aligned}(x+y)^3 &= 576-(x^2-y^2)^{\frac{3}{2}}\\ (x-y)^3 &= 72-(x^2-y^2)^{\frac{3}{2}}\end{aligned}\right\}$$

$$(x^2-y^2)^3=41472-648(x^2-y^2)^{\frac{3}{2}}+(x^2-y^2)^3,$$

$$\therefore\ (x^2-y^2)^{\frac{3}{2}}=\frac{41472}{648}=\frac{4608}{72}=\frac{512}{8}=64,$$

$$\left.\begin{aligned}\therefore\ (x+y)^3 &= 576-64=512\\ (x-y)^3 &= 8\end{aligned}\right\}\ldots\ldots(3).$$

$$\left.\begin{aligned}\therefore\ x+y &= 8\\ x-y &= 2\end{aligned}\right\}\ \therefore\ x=5 \text{ and } y=3.$$

Those who are skilled in the general Theory of Equations may find two pairs of imaginary simultaneous values of x and y from the equations (3).

5. Let x£. be the daily demand on A's Bank at first; then
y£. . . . . B's . . . .

$$3x + 3x \times 2 = \text{A's cash},$$
$$\text{and } 9x + 3y = \text{A's cash} + \text{B's},$$
$$\therefore \text{ by the question},$$
$$\left.\begin{array}{l} 3y = 7y - 4000 \\ \text{and } 9x + 3y = 7(x+y) \end{array}\right\},$$
$$\therefore y = 1000,$$
$$\text{and } x = 2000l.$$

6. Let x = sum paid for each small burner for 6 nights in the week, then the whole sum paid, as at first agreed, would have been $(3 + 5 + \frac{5}{4})\,x$. But the whole sum actually paid was

$$(3 + 5 + \tfrac{5}{4} + 2)\,\tfrac{7}{6}\,x,$$
$$\therefore \text{ by the question},$$
$$\frac{45}{24} \times 7x - \frac{37}{4} \cdot x = 1\,\frac{11}{20} = \frac{31}{20},$$
$$315x - 222x = 31 \times \frac{24}{20} = \frac{31 \times 6}{5},$$
$$\text{Hence, } x = \tfrac{2}{5}l.$$
$$\therefore \frac{45 \times 7}{24} \times \frac{2}{5} = \frac{63}{12} = \frac{21}{4} = 5l.\ 5s. \text{ the answer.}$$

7. Let x be the miles he travelled an hour at starting, $2y$ the distance from Cambridge to London.

Then, by the question,

$$\frac{y}{x} + y \div \tfrac{4}{5}x = 9 - \tfrac{1}{4} = \frac{35}{4},$$
$$\therefore \frac{y}{x} = 5 \;.\;.\;.\;.\;.\; (1).$$

Again, if z be the gain of the watch per hour, then during the actual 5 hours of riding the first half, as shown by (1), the watch will have gained $5z$, and he thinks he has been on the road $5 + 5z$ hours. Hence he supposes he has only $4 - 5z$ hours to ride y miles in, and, consequently, that his rate must

be $\dfrac{y}{4 - 5z}$ miles an hour; so that

$$\frac{y}{4 - 5z} = \tfrac{4}{3}\, x \;.\;.\;.\;.\; (2).$$

From the equations (1) and (2), we easily get

$$z = \tfrac{1}{20} \text{ hours.}$$

Lastly, the true time to the fourteenth mile-stone is $\dfrac{14}{x}$ hours; and, consequently, the apparent time would have been, on the second hypothesis,

$$\frac{14}{x} - \frac{1}{20} \times \frac{14}{x} \text{ or } \frac{133}{10\, x},$$

Hence, he would suppose he had $9 - \dfrac{133}{10x}$ hours to ride $y - 14$ miles in. He, therefore, would make his pace $= \dfrac{y - 14}{9 - \dfrac{133}{10x}}$ miles an hour; so that, by the question

$$\frac{14}{x} + (y - 14) \div \frac{y - 14}{9 - \dfrac{133}{10x}} = 9 + \frac{7}{60},$$

$$\text{or } \frac{140 - 133}{x} = \frac{7}{6},$$

$$\text{and } x = 6 \text{ miles},$$

$$\text{Hence, } 2\, y = 6 \times 10 = 60 \text{ miles.}$$

ST. JOHN'S COLLEGE, 1826.

1. The product = 0000 208; the quotent = 15000.

2. $\left(\frac{3}{2}\right)^{\frac{3}{2}} = \left(\frac{27}{8}\right)^{\frac{1}{2}} = \frac{1}{4} \cdot \sqrt{54} = 1.837$

   and $\sqrt{.012} = 109$, &c.

3. Ans. 4*l*. 16*s*.

4. Ans. 800*l*.

5. Answers

$$-\frac{2\sqrt{2}+3\sqrt{3}+2\sqrt{6}}{19} \text{ and } \frac{3\sqrt{2}+4-\sqrt{6}-2\sqrt{3}}{4},$$

6. The sum is $\dfrac{1}{a^4-x^4}$.

7. See (*Wood*, art. 111).

8. Dividing the first by the second,

$$\left(\frac{x}{y}\right)^{\alpha-\beta} = \frac{c}{d},$$

$$\therefore\ \frac{x}{y} = \left(\frac{c}{d}\right)^{\frac{1}{\alpha-\beta}},$$

$$\therefore \left(\frac{c}{d}\right)^{\frac{\alpha}{\alpha-\beta}} y^{\alpha+\beta} = a^\alpha b^\beta c,\ \therefore\ y^{\alpha+\beta} = a^\alpha b^\beta c \times \left(\frac{d}{c}\right)^{\frac{\alpha}{\alpha-\beta}},$$

$$\therefore\ y = \left(\frac{a^{\alpha^2-\alpha\beta} b^{\alpha\beta-\beta^2} d^\alpha}{c^\beta}\right)^{\frac{1}{\alpha^2-\beta^2}},$$

Hence, x may be found in a similar form.

9. If $\frac{1}{4}(x+2)+\frac{1}{3}x$ be $<\frac{1}{2}(x-4)+3$, and $>\frac{1}{2}(x+1)+\frac{1}{3}$,

then is $7x+6 < 6x+12$, and $> 6x+10$,

then $x<6$, and >4,

$\therefore\ x = 5$ is the integer value required.

10. Let x be the radix of the system; then

$$5+4x+x^2 = 290,$$

$$\therefore\ x = 15.$$

11. (1). Here the first term should be $\frac{2}{5}$.

The series is Geometric, its common ratio being $-\sqrt{\frac{5}{2}}$,

$$\therefore\ S = \frac{a(r^n-1)}{r-1} = \tfrac{2}{5}\cdot\frac{\left(-\sqrt{\frac{5}{2}}\right)^8-1}{-\sqrt{\frac{5}{2}}-1} = -\tfrac{2}{5}\,\frac{\left(\frac{5}{2}\right)^4-1}{\sqrt{\frac{5}{2}}+1},$$

$$= -\tfrac{2}{5}\{(\tfrac{5}{2})^2+1\}(\tfrac{5}{2}+1)(\sqrt{\tfrac{5}{2}}-1),$$

$$= -\tfrac{7}{5}\times\tfrac{29}{4}.(\sqrt{\tfrac{5}{2}}-1) = -\tfrac{203}{40}(\sqrt{10}-2),$$

which is easily exhibited in decimals.

(2). The common ratio is $\dfrac{1}{a-b}$,

$$\therefore\ S = (a^2-b^2 \times \frac{\left(\frac{1}{a-b}\right)^n - 1}{\frac{1}{a-b} - 1} = \frac{a^2-b^2}{(a-b)^{n-1}}.\frac{(a-b)^n-1}{a-b-1},$$

$$= \frac{a+b}{(a-b)^{n-2}}.\frac{(a-b)^n-1}{a-b-1}.$$

(3). Here the common difference is $-2\,ax$,

$$\therefore\ S = \{2\,a+(n-1)\,b\}\frac{n}{2},$$

$$= \{2\,(a+x)^2-(n-1)\,2\,ax\}\,\frac{n}{2} = \{(a+x)^2-(n-1)\,ax\}\,n,$$

$$= \{a^2+(3-n)\,ax+x^2\}\,n.$$

12. The Arithmetic mean $=\dfrac{a+b}{2}$; the Geometric $=\sqrt{(ab)}$, and, by the question,

$$\therefore \frac{a+b}{2} = 2\sqrt{(ab)},$$

$$\therefore \frac{a}{b} - 4\sqrt{\frac{a}{b}} = -1,$$

$$\therefore \sqrt{\frac{a}{b}} = 2\pm\sqrt{3},$$

$$\therefore \frac{a}{b} = 7\pm 4\sqrt{3} = (2\pm\sqrt{3})^2 = (2\pm\sqrt{3})(2\pm\sqrt{3}),$$

$$= \frac{2\pm\sqrt{3}}{2\mp\sqrt{3}}.$$

13. Ans. The roots are $2+\sqrt{3}$, and $\sqrt{(a^2-x^2)}+x$.

14. Ans. The G. C. M. is x^3-2x^2+x, and the reduced fraction is

$$\tfrac{2}{3}\cdot\frac{x^2+3}{x^2+x-1}.$$

ST JOHN'S COLLEGE, 1827.

1. First multiply by 39; reduce; then by 4, &c. $x = 11$.

2. Ans. $x = 16$; $y = 25$.

3. First $\dfrac{81}{16x-3}(3+4\sqrt{3x}) = 16x-8\sqrt{3x}+3,$

$$= (4\sqrt{x}-\sqrt{3})^2,$$

$$\text{or } \frac{81\sqrt{3}\,(4\sqrt{x}+\sqrt{3})}{16x-3} = (4\sqrt{x}-\sqrt{3})^2,$$

$$\therefore \frac{81\sqrt{3}}{4\sqrt{x}-\sqrt{3}} = (4\sqrt{x}-\sqrt{3})^2.$$

Hence, $4\sqrt{x} = \sqrt{3} + (81\sqrt{3})^{\frac{1}{3}} = \sqrt{3}+3\sqrt{3} = 4\sqrt{3},$

$$\therefore x = 3.$$

4. From the first,

$$3y^2 - 11 + 2x = (7+2y-y^2-x)^2,$$
$$= x^2 - 2x(7+2y-y^2) + (7+2y-y^2)^2,$$
$$\therefore x^2 - 2x(8+2y-y^2) = 3y^2 - 11 - (7+2y-y^2)^2,$$
$$\therefore x = 8+2y-y^2 \pm \sqrt{\{3y^2-11+(8+2y-y^2)^2 - (7+2y-y^2)^2\}},$$
$$= 8+2y-y^2 \pm (y+2),$$
$$= 10+3y-y^2 \text{ and } 6+y-y^2 \; \ldots\ldots \; (1).$$

Substituting these values in the second, we have

$$\sqrt{(y^2-3)} = \frac{10+4y-y^2}{10+2y-y^2},$$

This rationalized becomes

$$y^6 - 4y^5 - 20y^4 + 60y^3 + 152y^2 - 200y + 200 = 0.$$

But this cannot be resolved by the First Part of Algebra.

Again, substituting the second value of x, we have, in the second equation,

$$\sqrt{(y^2 + 2y + 1)} = \frac{6 + 2y - y^2}{6 - y^2}$$

$$\text{or } y + 1 = \frac{y^2 - 2y + 6}{y^2 - 6}.$$

$$\therefore\ y^3 + y^2 - 6y + 6 = y^2 - 2y + 6,$$

$$y^3 - 4y = 0,$$

$$\therefore\ y = 0, \text{ and } = 2, \text{ and } = -2,$$

$$\therefore\ x = 6, \text{ and } = 4, \text{ and } = 0.$$

The simultaneous values $\left.\begin{matrix} x = 6 \\ y = 0 \end{matrix}\right\} \left.\begin{matrix} x = 4 \\ y = 2 \end{matrix}\right\}$ satisfy the given equations when the sign of the symbol $\sqrt{}$ is considered $+$ and $\left.\begin{matrix} x = 0 \\ y = -2 \end{matrix}\right\}$ belong to those equations, when the radical is taken negative.

5. Let x be the number of balls in the large bag,
y that in the smaller,
z the number in a handful; then, by the question,

$$\left.\begin{aligned} x - z &= (y - z)^3 \\ x - z &= z^2 \\ \text{and } y - z + (y - z)^2 &= y + \tfrac{2}{3}y \end{aligned}\right\} \therefore \left.\begin{aligned} (y - z)^3 &= z^2 \\ (y - z)^2 &= z + \tfrac{2}{3}y \end{aligned}\right\}.$$

Let $y = uz$; then

$$\left.\begin{aligned} (u - 1)^3 z &= 1 \\ (u - 1)^2 z &= 1 + \tfrac{2}{3}u \end{aligned}\right\},$$

$$u - 1 = \frac{1}{1 + \frac{2}{3}u},$$

$$\therefore\ \tfrac{2}{3}u^2 + u - 1 - \tfrac{2}{3}u = 1,$$

$$\therefore \frac{2u^2}{3} + \tfrac{1}{3}u = 2,$$

$$\therefore\ u^2 + \tfrac{1}{2}u = 3,$$

$$u = -\tfrac{1}{4} \pm \sqrt{\frac{19}{16}} = \frac{-1 + \sqrt{19}}{4},$$

whence x, y, z, are easily found.

6. Suppose the stream to run from A to B, and make x miles the rate per hour at which one waterman would row in still water; also let y be the distance between the two towns; then the rate at which he proceeds from A to B, is $x + 4$,

. . . . B to A is $x - 4$,

$\therefore$ the time of the first voyage $= \dfrac{y}{x+4} + \dfrac{y}{x-4}$,

Also, had the water been still, the time would have been $\dfrac{2y}{x}$,

$\therefore$ by the question

$$\frac{y}{x+4} + \frac{y}{x-4} = \frac{2y}{x} + \frac{39}{60} \; . \; . \; . \; . \; . \; (1).$$

Again, the rate at which they go down the second day is $x + \dfrac{x}{2} + 4$, and the rate of returning is $x + \dfrac{x}{2} - 4$: and had the water been still the rate would have been $\dfrac{3x}{2}$; $\therefore$ by the question

$$\frac{y}{\frac{3x}{2}+4} + \frac{y}{\frac{3x}{2}-4} = \frac{2y}{\frac{3x}{2}} + \frac{8}{60} \; . \; . \; . \; . \; (2).$$

Equations (1) and (2) become

$$\left.\begin{aligned} \frac{x^3 - 16x}{32y} &= \frac{20}{13} \\ \frac{9x^3 - 64x}{32y} &= 30 \end{aligned}\right\},$$

$$\therefore\ 32y = \tfrac{104}{45}\, x^3,$$

$$\frac{9x^2 - 64}{x^2 - 16} = 13,$$

which gives $x = 6$.

7. Let t = actual time = time occupied by B, since, by supposition, he is the slowest workman, $t - x$ = time occupied by B.

Let y = work done by A in one month,

$z =$ . . . . . . B . . . . . .

$\therefore\ (t - x)\, y =$ A's actual work,

$t\, z \quad =$ B's . . . .

$$\left.\begin{aligned} \therefore (t-x)\,y + t\,z &= \text{whole length of foundation} \\ &= \tfrac{4}{5}\,t\,(y+z) \text{ by hyp:} \\ \text{Also, } \tfrac{4}{5}\,t\,z &= 3\,y \\ \text{and } \tfrac{4}{5}\,t\,y &= 12\,z = (t-x)\,y + 36 \end{aligned}\right\}, \quad \begin{aligned} &\ldots (1) \\ & \\ &\ldots (2) \\ &\ldots (3) \end{aligned}$$

$$\therefore \tfrac{4}{5}t\,(y+z) = 12z + 3\,y \text{ by addition,}$$

$$t\,z = \tfrac{15}{4}\,y,$$

$$(t-x)\,y = 12\,z - 36,$$

$\therefore$ substituting these values in the first equation,

$$12\,z - 36 + \tfrac{15}{4}\,y = 12\,z + 3\,y,$$

$$\therefore \tfrac{3}{4}y = 36,$$

$$\therefore \quad y = 48,$$

$\therefore$ substituting for y, in the remaining equations, we have

$$\left.\begin{aligned} \tfrac{4}{5} \times 48\,t &= 12\,z \\ 12\,z &= 48\,(t-x) + 36 \\ \therefore \tfrac{4}{5}\,t\,z &= 144 \end{aligned}\right\},$$

$$\left.\begin{aligned} \therefore\ 16\,t &= 5\,z \\ z &= 4\,(t-x) + 3 \\ t\,z &= 180 \end{aligned}\right\},$$

from the 1st of these,

$$z = \tfrac{16}{5}\,t,$$

$\therefore$ substituting in the 3rd,

$$\tfrac{16}{5}\,t^2 = 180$$

$$\therefore t^2 = \tfrac{225}{4}$$

$$\left.\begin{aligned} \therefore\ t &= \tfrac{15}{2} = 7\tfrac{1}{2} \\ \therefore\ z &= \tfrac{16}{5}t = 24 \end{aligned}\right\},$$

$\therefore$ substituting these values in the 2nd equation,

$$24 = 30 - 4\,x + 3$$

$$\therefore x = \tfrac{9}{4} = 2\tfrac{1}{4}$$

$$\therefore t - x = 5\tfrac{1}{4}$$

$$\begin{aligned} \therefore \text{ length of foundation} &= (t-x)\,y + tz \\ &= \tfrac{21}{4} \times 48 + \tfrac{15}{2} \times 24 \\ &= 252 + 180 \\ &= 432 \text{ yards.} \end{aligned}$$

ST JOHN'S COLLEGE, 1827.

1. Let x be the number of yards required; then, $\because$

$$\text{side} = \frac{\text{diagonal}}{\sqrt{2}},$$

$$\therefore \text{ square yards in the floor} = \frac{(\text{diagonal})^2}{2},$$

$$= \frac{(5+\frac{2}{3}+\frac{10}{36})^2}{2} = \left(\frac{214}{36}\right)^2 \times \tfrac{1}{2} = \left(\frac{107}{18}\right)^2 \times \tfrac{1}{2},$$

$\therefore$ by the question,

$$x \times \tfrac{3}{4} = \left(\frac{107}{18}\right)^2 \times \tfrac{1}{2},$$

$$\text{and } x = \tfrac{2}{3} \times \left(\frac{107}{18}\right)^2 = \frac{11449}{486},$$

$$= 23 \text{ yards, } 1 \text{ foot, } 8\tfrac{2}{27} \text{ inches.}$$

2. $\sqrt{\dfrac{5.04}{.012}} = \sqrt{420} = 20.4939015,$

also $\sqrt[4]{(97+28\sqrt{12})} = \sqrt{\{\sqrt{(97+28\sqrt{12})}} = 2+\sqrt{3}.$

3. Ans. Rate per cent. is $5\frac{5}{9}$.

4. Ans. $15'.\ 36''\frac{7}{48}$. past 10 o'clock.

5. The sum is $\dfrac{x^3-2x^2+3x-4}{x^4-x^3+x^2-x+1}$,

Also, assume $\dfrac{x+2}{x^3-x} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$;

clear it of fractions; collect coefficients of the powers of x, and put each of them $= 0$; the equations thus got will give

$$A = -2,\ B = \tfrac{1}{2},\ C = \tfrac{3}{2},$$

$$\text{and } \therefore \frac{x+2}{x^3-x} = -\frac{2}{x} + \frac{1}{2}\,\frac{1}{x+1} + \frac{3}{2}\cdot\frac{1}{x-1}.$$

6. See *Wood*, art. (76).

7. The quotient is $x^{\frac{1}{4}} + \frac{1}{2}\, a^{\frac{1}{2}}\, x^{-\frac{1}{4}} + \frac{1}{4}\, a\, x^{-\frac{3}{4}} +$ &c.
The coefficient of the ninth term is $-\, b\, (\beta + \gamma)$.

8. Their G. C. M. is x^2+1, and $\therefore$ the L. C. M. is x^4-1.

9. The common ratio of the first series being $-\frac{1}{x}$ if $\frac{1}{x}$ *be a proper fraction,* the sum *ad infra* is $\frac{x^4}{1+x}$.
The common difference of the second series is $-a+b$,

$$\text{and sum } = \{(n+1)\, a + (n-3)\, b\}\, \frac{n}{2}.$$

The Arithmetic means are $\frac{13}{16}$, $\frac{5}{8}$, $\frac{7}{16}$, $\frac{1}{4}$, $\frac{1}{16}$, $-\frac{1}{8}$, $-\frac{5}{16}$.

10. The roots of the first equation are $x = \frac{1}{a^2}$ and $\frac{1}{4a^2}$,

$$\text{of the second } \left.\begin{array}{l} x = .0\,2 \\ y = 2\,.\,9 \end{array}\right\},$$

$$\text{of the third} \quad z = 9 \text{ and } 4,$$

the third being solved by first clearing it of fractions and adding $x + 1$ to both sides, thus producing

$$x^2 + 2x + 1 = 49 + 14\sqrt{x} + x,$$

11. (1). Ans. $a^n + na^{n-1}\, b\sqrt{-1} - n.\, \frac{n-1}{2}\, a^{n-2}\, b^2 -$

$$\frac{n\,(n-1)\,(n-2)}{1.\ 2.\ 3}\, a^{n-3}\, b^3 \sqrt{-1},$$

and the n^{th} term, if $\frac{1}{\sqrt{(1-x^2)}}$, is $\frac{1.\,3.\,5 \;\ldots\; (2n-3)}{2.\,4.\,6 \;\ldots\; (2n-2)}\, x^{2n-2}$,

(2). The n^{th} term of $\frac{3.\ 5}{1^2.\ 3^2} +$ &c. is $\frac{(2n+1)\,(2n+3)}{n^2\,(n+2)^2}$,
and putting $n = 0,\ 1,\ 2$ &c. the series will be

$$-35,\ -\tfrac{32}{5},\ -\tfrac{36}{5},\ \text{\&c.}$$

12. From the given logarithms it is found that

$$\log 2 = 3010275 \text{ (which is not accurate)},$$

$$\begin{aligned} \text{and thence } \log 50000 &= \log 1000\,00 - \log 2, \\ &= 5 - .30\,10275, \\ &= 4.\,6989725 \text{ nearly.} \end{aligned}$$

13. Ans. 7.

14. Assume $\dfrac{1-3x+2x^2}{1-x+x^2} = 1 + Bx + Cx^2 + \&c.$

Multiply by denominator, collect coefficients of x°, x, x^2, &c. and put them each $= 0$. From the equations thus made, we easily get

$$B = -2,\ C = -1, \&c.$$

Observe, the same series will be produced by actual division.

ST. JOHN'S COLLEGE, 1828.

1. Multiply by 24, reduce; then multiply by $13 - 2x$ and get $x = \frac{3}{2}$.

2. First, $bc\,x - x^2\,y = xy^2 - acy$,

$$c\,(bx + ay) = xy\,(x + y)\ .\ .\ .\ .\ .\ .\ .\ (1).$$

also, from the second,

$$axy^2 + bx^2y - x^2y^3 = abc\,(x + y) - abc^2,$$

$$\therefore\ xy\,(bx + ay) = x^2y^2 + abc\,(x + y) - abc^2\ .\ .\ (2).$$

$$\therefore\ xy\,\frac{xy}{c}.(x + y) = x^2y^2 + abc\,(x + y) - abc^2,$$

$$x^2y^2\,(x + y) = cx^2y^2 + abc^2\,(x + y) - abc^3,$$

$$x^2y^2\,(x + y - c) = abc^2\,(x + y) - abc^3,$$

$$= abc^2\,(x + y - c)$$

$$\left.\begin{aligned}\therefore\ x^2y^2 &= abc^2\\ \text{and } x + y &= c\end{aligned}\right\}\ \begin{aligned}&.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (3).\\ &.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (4).\end{aligned}$$

Combining (3) and (1), we get

$$c\left(bx \pm \frac{ac\sqrt{ab}}{x}\right) = \pm\,c\sqrt{(ab)}\left(x \pm \frac{c\sqrt{ab}}{x}\right)$$

$$\therefore\ bx^2 \pm ac\sqrt{ab} = \pm\sqrt{ab}\,x^2 + abc,$$

$$\therefore\ x^2\,(b \mp \sqrt{ab}) = abc \mp ac\sqrt{ab},$$

$$x^2 = \frac{abc \mp ac\sqrt{ab}}{b \mp \sqrt{ab}} = ac\frac{\sqrt{b} \mp \sqrt{a}}{\sqrt{b} \mp \sqrt{a}} = ac,$$

$$\therefore\ x = \pm\sqrt{(ac)},$$

$$\text{Hence, } y^2 = \frac{abc^2}{ac} = bc,$$

$$y = \pm \sqrt{(bc)}.$$

which are two pairs of simultaneous values.

Again, combining (4) and (1), we have

$$c\,\{bx + a(c-x)\} = x(c-x)\,c,$$

$$\therefore\ bx + ac - ax = cx - x^2,$$

$$\therefore\ x^2 - (a-b+c)\,x = -ac,$$

$$\therefore\ x = \frac{a-b+c}{2} \pm \sqrt{\frac{(a-b+c)^2 - 4ac}{4}}$$

$$= \frac{a-b-c \pm \sqrt{(a^2+b^2+c^2-2ab-2ac-2bc)}}{2},$$

$$\text{and } \therefore\ y = c - x$$

$$= \frac{-a+b+3c \pm \sqrt{(a^2+b^2+c^2-2ab-2ac-2bc)}}{2}.$$

3. First, $(4x^2+8)^{\frac{3}{2}} + \dfrac{3}{\sqrt{(4x^2+8)}} = 16x^2+24,$

then, squaring

$$(4x^2+8)^3 + 6.(4x^2+8) + \frac{9}{4x^2+8} = 256x^4+768x^2+576,$$

$$\text{or, } 64x^6+384x^4+768x^2+512+24x^2+48+\frac{9}{4x^2+8},$$

$$\therefore\ 64x^6+128x^4+24x^2+\frac{9}{4x^2+8} = 46$$

$$256x^8+512x^6+96x^4-64x^2$$
$$512x^6+1924x^4+192x^2-128+9=0,$$

$$\therefore\ 256x^8+1024x^6+1120x^4+128x^2-119=0,$$

$$\text{or, } (2x)^8+16(2x)^6+70(2x)^4+32.(2x)^2-119=0,$$

$$\therefore 119\,\{(2x)^2-1\}+70(2x)^4-87(2x)^2+(2x)^8+16(2x)^6=0,$$

$$\therefore\ 119\,\{(2x)^2-1\}+87\,\{(2x)^2-1\}-17(2x)^4+16(2x)^6$$
$$+(2x)^8=0,$$

$$\therefore\ 119\,\{(2x)^2-1\}+87(2x)^2\,\{(2x)^2-1\}+17^8\,2(2x)^4$$
$$\{(2x)^2-1\}+(2x)^6\,\{(2x)^2-1\}=0,$$

$$\therefore\ \{(2x)^2-1\}\,\{(2x)^6+17(2x)^4+87(2x)^2+119\}=0,$$

$$\therefore\ (2x)^2-1=0,$$

$$\therefore\ x = \pm\tfrac{1}{2}.$$

The principles of the Second Part of Algebra would give six other roots of the equation.

4. From the second we have

$$\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\left\{3x^{\frac{4}{3}}\,y^{\frac{2}{3}}+\frac{91}{216}\,x^{\frac{2}{3}}\sqrt{(x^{\frac{2}{3}}+y^{\frac{2}{3}})}\right\}=(\tfrac{5}{6})^3-x^2-y^2,$$

$$\therefore (x^{\frac{2}{3}}+y^{\frac{2}{3}})\;\left\{3x^{\frac{2}{3}}\,y^{\frac{2}{3}}+\frac{91}{216}\sqrt{(x^{\frac{2}{3}}+y^{\frac{2}{3}})}\right\}=(\tfrac{5}{6})^3-(x^2+y^2)$$

$$=(\tfrac{5}{6})^3-(x^{\frac{2}{3}}+y^{\frac{2}{3}})\;(x^{\frac{4}{3}}-x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}\},$$

$$\therefore (x^{\frac{2}{3}}+y^{\frac{2}{3}})\left\{x^{\frac{4}{3}}+2\,x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}+\frac{91}{216}\sqrt{(x^{\frac{2}{3}}+y^{\frac{2}{3}})}\right\}=(\tfrac{5}{6})^3$$

$$\therefore (x^{\frac{2}{3}}+y^{\frac{2}{3}})\left\{(x^{\frac{2}{3}}+y^{\frac{2}{3}})^2+\frac{91}{216}\sqrt{(x^{\frac{2}{3}}+y^{\frac{2}{3}})}\right\}=(\tfrac{5}{6})^3,$$

$$\therefore (x^{\frac{2}{3}}+y^{\frac{2}{3}})^3+\frac{91}{216}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=(\tfrac{5}{6})^3,$$

$$\therefore (x^{\frac{2}{3}}+y^{\frac{2}{3}})=-\frac{91}{432}\pm\sqrt{\left\{\left(\frac{91}{432}\right)^2+(\tfrac{5}{6})^3\right\}},$$

$$=\frac{-91\pm\sqrt{\{8281+108000\}}}{432},$$

$$=\frac{-91\pm 341}{432},$$

$$=\frac{250}{432}\text{ and }-\frac{432}{432},$$

$$=\frac{125}{216}\text{ and }-1,$$

$$=(\tfrac{5}{6})^3\text{ and }-1,$$

$$\therefore\;(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{1}{2}}=\tfrac{5}{6}\text{ and }-1\;.\;.\;.\;.\;.\;(1).$$

Substituting the first of these values in the first of the given equations, we get

$$\frac{25}{(xy)^{\frac{2}{3}}}+48\,(xy)^{\frac{1}{3}}=241.$$

$\therefore$ putting $u=xy$,

$\therefore\ 48\,u-241\,u^{\frac{2}{3}}+25=0,$

$\therefore\ 2\,(27\,u-1)-6\,u-241\,u^{\frac{2}{3}}+27=0,$

$\therefore\ 2\,(27\,u-1)-6\,u-27\times 9\,u^{\frac{2}{3}}+2\,u^{\frac{2}{3}}+27=0,$

$\therefore\ 2\,(27\,u-1)-27\,(9\,u^{\frac{2}{3}}-1)-2\,u^{\frac{2}{3}}\,(3\,u^{\frac{1}{3}}-1)=0.$

$\therefore\ (3u^{\frac{1}{3}}-1)(18u^{\frac{2}{3}}+6u^{\frac{1}{3}}+2-81u^{\frac{1}{3}}-27-2u^{\frac{2}{3}})=0,$

$\therefore\ (3u^{\frac{1}{3}}-1)(16u^{\frac{2}{3}}-75u^{\frac{1}{3}}-25)=0,$

$\therefore\ 3u^{\frac{1}{3}}-1=0\ \ldots\ldots\ldots\ldots\ (2),$

and $16u^{\frac{2}{3}}-75u^{\frac{1}{3}}-25=0\ \ldots\ldots\ (3).$

Equation (2) gives $u=\frac{1}{27}$, and (3) gives $u=125$, or $-\frac{125}{256}$.

Combining $xy=\frac{1}{27}$ with $x^{\frac{2}{3}}+y^{\frac{2}{3}}=\frac{25}{36}$ it will easily be found that $x=\frac{8}{27}$ and $y=\frac{1}{8}$. Other simultaneous values may be obtained by combining the other values of u with $x^{\frac{2}{3}}+y^{\frac{2}{3}}=\frac{25}{36}$.

Also, other simultaneous values may be found by substituting $(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{1}{2}}=-1$ in the first of the given equations.

5. Let x o'clock be the hour they are striking; then the xth stroke of the faster clock is at $3x-2$ whilst the xth stroke of the slower is at $4x-1$.

Let the yth stroke of the faster coincide with the zth stroke of the slower; then

$$3y-2=4z-1.$$

Of this the solutions in integers are

$$z=2,\ 5,\ 8,\ 11,\ \&c.$$
$$y=3,\ 7,\ 11,\ 15,\ \&c.$$

$\therefore$ by the question, the coincidences are 3,

$$\text{and } 2x-3=19,$$
$$\therefore\ x=11.$$

6. First, $5\frac{1}{3}$ points $=\frac{1}{6}$ of 32 points $=\frac{1}{6}\times 360°=60°$,
and $2\frac{2}{3}\ \ldots\ldots=\frac{1}{12}$ of 32 $\ldots\ldots=\frac{1}{12}\times 360°=30°$.

Also, before the first tack, the distance of the vessels is

$$3q+\frac{np}{2}-\frac{p}{2}, \text{ or } 3q+\frac{(n-1)p}{2}.$$

Again, after the first tack, the distance of the cutter from the first direction is

$$p \sin 60^\circ \text{ or } p\frac{\sqrt{3}}{2},$$

that of the smuggler is $np \sin 30^\circ$, or $\frac{np}{2}$.

Hence, (by 47th prop. of Euclid, b. i.) the distance of the vessels, at the end of the first tack, is

$$d_1 = \sqrt{\left\{\left(\frac{np+p\sqrt{3}}{2}\right)^2 + \left(3q+\frac{n-1}{2}\cdot p + np\frac{\sqrt{3}}{2} - \frac{p}{2}\right)^2\right\}}$$

The vessels now changing their tacks, but still preserving the same angular distance from windward, at the end of the second tack, they lie in the very same line as at first, viz. to windward; and this will evidently be the case also at the end of the 4th, 6th, &c. tacks. Their distance is now

$$d_2 = 3q + \frac{n-1}{2}p - p(1-n\sqrt{3}).$$

Similarly,

$$d_4 = 3q + \frac{n-1}{2}p - 2p(1-n\sqrt{3}) \text{ the distance after the 4th tack}$$

$$d_6 = 3q + \frac{n-1}{2}p - 3p(1-n\sqrt{3}) \text{ . . . . 6th . .}$$

&c.

Also,

$$d_3 = \sqrt{\left[\left(\frac{np+p\sqrt{3}}{2}\right)^2 + \left\{3q+\frac{n-1}{2}p - p(1-n\sqrt{3}) + \frac{np\sqrt{3}}{2} - \frac{p}{2}\right\}^2\right]},$$

$$d_5 = \sqrt{\left[\left(\frac{np+p\sqrt{3}}{2}\right)^2 + \left\{3q+\frac{p}{2}(n+n\sqrt{3}-2) - 2p(1-n\sqrt{3})\right\}^2\right]},$$

$$d_7 = \sqrt{\left[\left(\frac{np+p\sqrt{3}}{2}\right)^2 + \left\{3q+\frac{p}{2}(n+n\sqrt{3}-2) - 3p(1-n\sqrt{3})\right\}^2\right]},$$

&c.

Now, if the cutter come within shot of the smuggler in an odd tack, let that tack be the $(2x+1)^{\text{th}}$; but if at the end of an even tack, let it be the $2y^{\text{th}}$; then

$$\left.\begin{aligned} r = d_{2y} &= 3q + \frac{n-1}{2}p - yp\,(1-n\sqrt{3}, \\ \text{or } r^2 = d^2_{2x+1} &= \left(\frac{np+p\sqrt{3}}{2}\right)^2 + \left\{3q + \frac{p}{2}(n+n\sqrt{3}-2)\right. \\ &\qquad \left. - xp\,(1-n\sqrt{3})\right\}^2, \end{aligned}\right\}$$

whence x or y may easily be found, and we get

$$y = \frac{6q-2r+(n-1)p}{2p\,(1-n\sqrt{3})},$$

$$\text{or, } x = \frac{6q+p\,(n+n\sqrt{3}-2) \mp \sqrt{\{4r^2-p^2(n+\sqrt{3})^2\}}}{2p\,(1-n\sqrt{3})},$$

and y or x must, in specific cases, be found in nearest integers.

7. Let x minutes be the time of filling in the second irruption,
y feet the length of the tunnel and shaft at that time,
z the breadth and depth of each tunnel;

then the area of a horizontal section of the two tunnels and the bottom of the shaft is

$$2yz \text{ square feet,}$$

and the area of a section of the shaft is $4z^2$.

Also, at the second irruption, the volume of the tunnels and shaft was

$$2yz \times z + 4z^2 \times 4z, \text{ or } 2yz^2 + 16z^3 \text{ cubic inches,}$$

and the time of filling, the first time, is $x-10$ minutes.

Again, let v be the velocity of the water, or number of cubic inches per minute entering at the first irruption, and v' that at the second; then, since

volume filled $\propto$ vel. $\times$ time,

volume, at first, : volume, at the second irruption, :: $v \times (x-6) : v' \times x$,

$$\therefore \text{ volume, at first, } = \frac{v\,(x-10)}{v'\,x}.\ (2yz^2+16z^3),$$

and $\therefore$ length of the tunnels and shaft, at first,

$$= \frac{v\,(x-10)}{v'\,x}\,(y+8\,z) - 8\,z.$$

Now, time of filling $2\,y\,z^2+16\,z^3$ with $v = \dfrac{2\,y\,z^2+16\,z^3}{v}$,

and the time of filling $\dfrac{v\,(x-10)}{v'\,x}\,(2\,yz^2+16\,z^3)$ with

$$v = \frac{x-10}{v'\,x}\,(2\,y\,z^2+16\,z^3),$$

$\therefore$ by the question,

$$\frac{2y\,z^2+16\,z^3}{v} = \frac{3}{2}\cdot\frac{x-10}{v'\,x}\,(2\,y\,z^2+16\,z^3),$$

$$\text{and } x = \frac{30\,v}{3\,v-2\,v'} \quad \ldots\ldots \quad (1).$$

Again, velocity of ascending surface $\propto \dfrac{\text{velocity of influx}}{\text{horizontal section}}$,

$$\therefore \frac{v'}{2yz} : \frac{v}{4\,z^2} :: \text{velocity of the levels} : \text{vel. in the shaft},$$

$$:: 1 : 8,$$

$$\therefore \frac{z}{y} = \frac{v}{16\,v'} \quad \ldots\ldots\ldots \quad (2).$$

Again, had the tunnels been 110 feet longer, the horizontal section of the levels would have been

$$2\,(y+110)\,z \text{ square feet,}$$

and the ascending velocities, at the first irruption, the second, and for this supposed section, would have been respectively

$$\frac{v'\,x}{2z(x-10)\,(2yz^2+16z^3)},\ \frac{v'}{2\,y\,z},\ \frac{v'}{2\,(y+110)\,z},$$

and the velocities of ascent in the shaft, in the first and second irruptions, are as

$$\frac{v}{4\,z^2} \text{ and } \frac{v'}{4z^2},$$

$\therefore$ by the question,

$$\frac{v'\,x}{2z\,(x-10)\,(2y\,z^2+16\,z^3)} - \frac{v'}{2\,y\,z} = \frac{1}{9}\cdot\frac{v-v'}{4\,z^2},$$

whence $$\frac{x}{z\,(x-10)\,(y+8\,z)} - \frac{2\,z}{y} = \frac{1}{9}\left(\frac{v}{v'} - 1\right) \;.\;.\; (3).$$

Also, $$\frac{v'}{2\,y\,z} - \frac{v'}{2\,(y+110)\,z} = \frac{1}{9}\cdot\frac{v-v'}{4\,z^2},$$

$$\therefore \frac{z}{y} - \frac{z}{y+110} = \frac{1}{18}\cdot\left(\frac{v}{v'} - 1\right),$$

$$\therefore \frac{1980\,z}{y\,(y+110)} + 1 = \frac{v}{v'} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (4).$$

To solve these four equations, make $u = \dfrac{v'}{v}$; then they become

$$\left.\begin{aligned} x &= \frac{30}{3-2\,u} &\;.\;.\;.\; (5).\\ \frac{y}{z} &= 16\,u &\;.\;.\;.\; (6).\\ \frac{x}{z\,(x-10)\,(y+8\,z)} - \frac{2\,z}{y} &= \frac{1}{9}\cdot\left(\frac{1}{u} - 1\right) &\;.\;.\;.\; (7).\\ \frac{1980\,z}{y\,(y+110)} + 1 &= \frac{1}{u} &\;.\;.\;.\; (8).\end{aligned}\right\}$$

From the (6) we get $u = \dfrac{y}{16z}$, and substituting this in (5), (7), (8), we get

$$\left.\begin{aligned} x &= \frac{240\,z}{24\,z-y} &\;.\;.\;.\; (9).\\ \frac{9\,xy}{z\,(x-10)\,(y+8z)} &= 34\,z+y &\;.\;.\;(10).\\ 260\,z-16\,yz+y\,(y+110) &= 0 &\;.\;.\;(11).\end{aligned}\right\}$$

From (9), $x = \dfrac{240\,z}{24z-y}$, which being put in (10), gives

$$\left.\begin{aligned} \frac{216}{y+8z} &= 34\,z + y &\;.\;.\;(12).\\ \text{also, } z &= \frac{y\,(y+110)}{16\,y-260}, &\;.\;.\;(13).\end{aligned}\right\}$$

whence x, y, z, and u may be found.

ST. JOHN'S COLLEGE, 1828.

1. The quotient is 5020.

$$\sqrt{(1+\sqrt{-12})} = \sqrt{\frac{\sqrt{13}+1}{2}} + \sqrt{\frac{\sqrt{13}-1}{2}} \cdot \sqrt{-1},$$

and $\sqrt{(6+\sqrt{8}-\sqrt{12}-\sqrt{24})} = \sqrt{3}-\sqrt{2}-1.$

This third is done, by assuming

$$\sqrt{(6+\sqrt{8}-\sqrt{12}-\sqrt{24})} = \sqrt{x}+\sqrt{y}+\sqrt{z},$$

then, $6+\sqrt{8}-\sqrt{12}-\sqrt{24} = x+y+z+2\sqrt{(xy)}+2\sqrt{(xz)} + 2\sqrt{(yz)},$

$$\therefore \sqrt{(xy)}+\sqrt{(xz)}+\sqrt{(yz)} = \sqrt{2}-\sqrt{3}-\sqrt{6},$$

$$\therefore \sqrt{(xy)} = \sqrt{2},\ \sqrt{(xz)} = -\sqrt{3},\ \sqrt{(yz)} = -\sqrt{6},$$

$$\therefore xyz = 36, \&c.$$

Again, $2+\sqrt{3} = 2+1.7320508 = 3.7320508,$

and $\therefore \sqrt[3]{(2+\sqrt{3})} = 1.55$, &c.

2. The five quarters of wheat are worth

$$\frac{32}{33} \times 60 \times 5 = \frac{160}{11} \times 20 \text{ shillings},$$

and the barley . . . .

$$\frac{32}{33} \times 54 \times 3 = \frac{32 \times 54}{11} \text{ shillings.}$$

Ans. 22*l*. 8*s*.

3. Let x be the pounds he had in the 3 per cents; then the pounds stock he gets in the 4 per cents are $\frac{100}{103\frac{5}{8}}x$ or $\frac{800}{829}x$.

The income on the cash, when in the 3 per cents, was $\frac{100}{84\frac{1}{4}}x \times 3$,

$\therefore$ by the question,

$$\frac{3200}{829}x - \frac{1200}{337}x = 10.$$

$$\therefore x\,(337 \times 32 - 829 \times 12) = 829 \times 337,$$

$$\therefore x = \frac{829 \times 337}{836},$$

which is easily reduced.

4. First, add the two last fractions.

$$\text{Ans. } \frac{3x^2}{x^3-1}.$$

The quotient is $x^{-\frac{1}{2}} - \frac{1}{2}\, x^{-\frac{3}{2}} - \frac{1}{2}\, x^{-\frac{5}{2}} - \&c.$

Also, $\dfrac{1}{x^4-1} = \frac{1}{4}.\dfrac{1}{x-1} - \frac{1}{4}.\dfrac{1}{x+1} - \frac{1}{2}.\dfrac{1}{x^2+1}.$

This is done by the method of indeterminate coefficients; first assuming

$$\frac{1}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1}.$$

5. $\sqrt{\frac{1}{2}}$ is $<$ $(\frac{2}{3})^{\frac{2}{3}}$, $\sqrt{2}+\sqrt{7}$ is $>$ $\sqrt{3}+\sqrt{5}$,

and $a+\sqrt{a}$ is $<$ $1+a^{\frac{3}{2}}$.

6. Ans. G. C. M. $= x^2-1$.

7.
$$\begin{aligned} \text{Let } S &= 2.34545\ldots\ldots \\ \therefore\ 1000\,S &= 2345.4545\ldots\ldots \\ \text{and } 10\,S &= 23.4545\ldots\ldots \\ \therefore\ S &= \frac{2322}{990} = \frac{1161}{495} = \frac{129}{55}. \end{aligned}$$

Again, $1 - \frac{1}{2} + \ldots\ldots = \dfrac{a}{1-r} = \dfrac{1}{1+\frac{1}{2}} = \frac{2}{3}$,

and $\dfrac{n-1}{n} + \dfrac{n-2}{n} + \ldots\, n$ terms $= \dfrac{n-1}{2}.$

Finally, let x, y, be the two quantities; then

$$\frac{\sqrt{(xy)}}{\dfrac{x+y}{2}} \text{ is given } = 2a, \text{ suppose;}$$

$$\text{or } \frac{x}{y} - \frac{1}{a}.\sqrt{\frac{x}{y}} = -1,$$

$$\therefore \sqrt{\frac{x}{y}} = \frac{1}{2a} \pm \sqrt{\left(\frac{1}{4a^2} - 1\right)},$$

whence $\dfrac{x}{y}.$

8. (1). $x^{-3}+x^{-\frac{3}{2}}=2$,

$$\therefore\ x^{-\frac{3}{2}}=-\tfrac{1}{2}\pm\sqrt{\tfrac{9}{4}}=\frac{-1\pm 3}{2}=1 \text{ and } -2,$$

$$\therefore\ x=1 \text{ and } (\tfrac{1}{4})^{\frac{1}{3}}.$$

(2). $4x^2+6\sqrt{(1+x)}.\,2x=27\,(1+x)$,

$\therefore\ 2x+3\sqrt{(1+x)}=\pm\sqrt{\{36\,(1+x)\}}=\pm 6\sqrt{(1+x)}$,

$\therefore\ 2x=3\sqrt{(1+x)}$ and $-9\sqrt{(1+x)}$,

$\therefore\ 4x^2=9+9x$ and $81+81x$,

$\therefore\ x^2-\frac{9}{4}x=\frac{9}{4}$ and $x^2-\frac{81}{4}x=\frac{81}{4}$,

$$x=\frac{9\pm\sqrt{(81+144)}}{8} \text{ and } x=\frac{81\pm\sqrt{(81^2+16\times 81)}}{8},$$

$$\therefore\ x=\frac{9\pm 15}{8} \text{ and } x=\frac{81\pm 9\sqrt{97}}{8},$$

&c.

(3). Since $\dfrac{x}{1+\sqrt{(1+x)}}=\sqrt{(1+x)}-1$,

$$\therefore\ x-4=1+x+1-2\sqrt{(1+x)},$$

and $x=8$.

(4). From $\sqrt{x}\,(1-xy)=y-x$, we get

$$y=\frac{x+\sqrt{x}}{1+x\sqrt{x}}=\sqrt{x}.\,\frac{1+\sqrt{x}}{1+x^{\frac{3}{2}}}=\frac{\sqrt{x}}{1-\sqrt{x}+x}$$

Hence, $1-y=\dfrac{(1-\sqrt{x})^2}{1-\sqrt{x}+x}$,

and $1+y=\dfrac{1+x}{1-\sqrt{x}+x}$,

$$\left.\begin{aligned}\therefore\ \sqrt{(1-y^2)}&=\frac{(1-\sqrt{x})\sqrt{(1+x)}}{1-\sqrt{x}+x}\\ \text{and } y-x&=\sqrt{x}\,\frac{(1-\sqrt{x})\,(1+x)}{1-\sqrt{x}+x}\end{aligned}\right\}.$$

Now, dividing the first equation by the second, we get

$$\frac{a}{\sqrt{x}}=\frac{x\sqrt{(1-y^2)}}{y-x}=x.\,\frac{(1-\sqrt{x})\sqrt{(1+x)}}{\sqrt{x}(1-\sqrt{x})\,(1+x)},$$

$$\therefore\ a\sqrt{(1+x)}=x,$$

$$\therefore\ x^2-a^2x=a^2,$$

$$\therefore x = \frac{a^2 \pm \sqrt{(a^4+4a^2)}}{2} = \frac{a^2 \pm a\sqrt{(a^2+4)}}{2}.$$

Hence, $y = \dfrac{x+\sqrt{x}}{1+x\sqrt{x}}$.

It also appears from

$$a(1-\sqrt{x})\sqrt{(1+x)} = x(1-\sqrt{x}),$$

that $\sqrt{x}-1 = 0$,

$$\left.\begin{aligned}\text{or, } x &= 1\\ \text{Hence, } y &= 1\end{aligned}\right\}.$$

9. Divide repeatedly 2304, in its own scale of 5 by 11.

$$\begin{array}{r|l} 11) & 2304 \\ \hline 11) & 104 - 10 \\ \hline & 2 - \ \ 7 \end{array}$$

$\therefore$ in the scale, whose radix is 11, the number is (t meaning the digit 10) 27 t.

To make the number consist of 1, 3, 3^2, &c. we must transform it to the system whose radix is 3.

$$\begin{array}{r|l} 3) & 2304 \\ \hline 3) & 414 - \ 2 \\ \hline 3) & 121 - \ 1 \\ \hline 3) & 22 - \ 0 \\ \hline & 4 - \ 0 \end{array}$$

$\therefore$ in the ternary system, the number is

40012,

and it $\therefore$ consists of 2×1, 3, 0×3^2, 0×3^3, 4×3^4.

10. Let x be the number of guineas, y of crowns; then

$$21x + 5y = 20000,$$

and this solved in integers gives

$$x = 5, 10, 15, 20 \ldots\ldots$$
$$y = 3979, 3958, 3937, 3916 \ldots\ldots$$

Also the number of solutions (*Barlow's* Theory of Numbers, p. 325), is the greatest integer in

$$\frac{20000}{5 \times 21}, \text{ or in } \frac{4000}{21}, \text{ or } 190.$$

11. For the rule, see *Barlow's* Theory of Numbers, p. 33.

$$\text{Since } 2160 = 2^4.\ 3^3.\ 5.$$
$$\begin{aligned}\text{Number of divisors} &= (4+1)(3+1)(1+1),\\ &= 40.\end{aligned}$$

12.

$$\left.\begin{aligned}&n.\,(n-1)\ldots(n-r+1) : n\,(n-1)\ldots(n-r+2) :: 10 : 1\\ \text{and } &\frac{n\,(n-1)\ldots(n-r+1)}{1.\ 2\ \ldots\ \ r} : \frac{n(n-1)\ldots(n-r+2)}{1.\ 2\ \ldots\ r-1} ::\ 5 : 3\end{aligned}\right\}$$

$$\left.\begin{aligned}\therefore\ n-r&=9\\ \text{and } \frac{n-r+1}{r} &= \frac{5}{3}\end{aligned}\right\}\therefore\ \begin{aligned}n &= 15\\ r &= \ 6.\end{aligned}$$

13. Amount $=$ PR^n (*Wood*, art. 397.)

$$\therefore \left(\frac{21}{20}\right)^n = \frac{1050}{100} = \frac{21}{20} \times 10,$$

$$\left(\frac{21}{20}\right)^{n-1} = 10,$$

$$\therefore\ n-1 = \frac{1}{\log 21 - \log 20}.$$

$$\begin{aligned}\text{But } \log 21 &= \log \tfrac{3}{2} \times \tfrac{5}{7} \times 10,\\ &= 1 + \log \tfrac{3}{2} + \log \tfrac{7}{5} = 1 + .17609,\\ &\qquad\qquad\qquad\qquad\quad + .14613,\\ &= 1.32222,\\ \text{and } \log 20 &= 1+\log 2 = 1+\tfrac{1}{4}\log 16 = 1+\tfrac{1}{4}\log\tfrac{16}{10}\times 10,\\ &= \tfrac{3}{4} + \tfrac{1}{4}\log 1.6 = \tfrac{5}{4} + .05103,\\ &= 1.30103,\\ \therefore \log 21 - \log 20 &= .0\,2119.\end{aligned}$$

$$\therefore\ n-1 = \frac{1}{0.02119} = \frac{100000}{2119},$$

$$\text{and } n = 48\frac{407}{2119} \text{ years.}$$

14. When x is small,

$$\text{it} = \frac{1+\frac{1}{2}x+1-\frac{2}{3}x}{1+x+1+\frac{1}{2}x} \text{ nearly} = \frac{2-\frac{1}{6}x}{2+\frac{3}{2}x} \text{ nearly}$$
$$= 1 - \tfrac{5}{6}x \text{ nearly by actual division.}$$

When x is very large $\dfrac{1}{x}$ is very small.

Let $\therefore\ u = \dfrac{1}{x}$; then

$$\frac{\dfrac{(1+u)^{\frac{1}{2}}}{\sqrt{u}} + \dfrac{(1-u)^{\frac{2}{3}}}{u^{\frac{2}{3}}}}{\dfrac{1+u}{u} + \dfrac{\sqrt{(1+u)}}{\sqrt{u}}} = \frac{\sqrt{u}\sqrt{(1+u)} + u^{\frac{1}{3}}(1-u)^{\frac{2}{3}}}{1+u+\sqrt{u}\sqrt{(1+u)}},$$

$$= \frac{\sqrt{u}.\,(1+\frac{1}{2}u-\frac{1}{8}u^2)+u^{\frac{1}{3}}(1-\frac{2}{3}u-\frac{1}{9}u^2)}{1+u+\sqrt{u}.\,(1+\frac{1}{2}u-\frac{1}{8}u^2)} \text{ nearly}$$

$$= \frac{u^{\frac{1}{3}} + u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{4}{3}} + \frac{1}{2}u^{\frac{3}{2}} - \frac{1}{9}u^{\frac{7}{3}} - \frac{1}{8}u^{\frac{5}{2}}}{1+u^{\frac{1}{2}}+u+\frac{1}{2}u^{\frac{3}{2}}-\frac{1}{8}u^{\frac{5}{2}}},$$

$$= u^{\frac{1}{3}} + u^{\frac{1}{2}} - u - \tfrac{5}{3}u^{\frac{4}{3}} + \tfrac{1}{2}u^{\frac{3}{2}} + \tfrac{2}{3}u^{\frac{11}{6}} \text{ nearly,}$$

the terms containing u^2, and the higher powers being omitted. Hence the value of the function, true up to small quantities of the order $\left(\dfrac{1}{x}\right)^2$ is

$$\frac{1}{x^{\frac{1}{3}}} + \frac{1}{\sqrt{x}} - \frac{1}{x} - \frac{5}{3x^{\frac{4}{3}}} + \tfrac{1}{2}.\,\frac{1}{x^{\frac{3}{2}}} + \tfrac{2}{3}.\,\frac{1}{x^{\frac{11}{6}}}.$$

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1. Multiply by 210, &c. Ans. $x = 4$.

2. Ans. $\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\} \left.\begin{matrix} x = 4 \\ y = 12 \end{matrix}\right\} \left.\begin{matrix} x = \frac{4}{17} \\ y = \frac{36}{19} \end{matrix}\right\}.$

3. First $\sqrt{x} + \dfrac{1}{\sqrt{x-2}} = 8\left(\dfrac{1}{x} + \dfrac{1}{\sqrt{x-2}}\right)$,

$$\therefore\ x - 2\sqrt{x} + 1 = \frac{8}{x}(x + \sqrt{x-2}),$$

$$\therefore\ (\sqrt{x}-1)^2 = \frac{8}{x}(\sqrt{x}-1)(\sqrt{x}+2),$$

$$(\sqrt{x}-1)\left\{\sqrt{x}-1-\frac{8(\sqrt{x}+2)}{x}\right\} = 0,$$

$$\therefore\ (\sqrt{x}-1)(x\sqrt{x}-x-8\sqrt{x}-16) = 0,$$

$$\left.\begin{array}{l}\therefore\ \sqrt{x}-1=0\\ \text{and } x\sqrt{x}-x-8\sqrt{x}-16\end{array}\right\}_{=0} \ldots\ldots\ldots \begin{array}{r}(1)\\(2)\end{array}$$

From (1) $x=1$; from (2) $x=16$. *But (2) is a cubic.*

4. From the first,

$$4+4y\sqrt{x}+xy^2 = x^2y+xy^2 + \frac{y^3}{4},$$

$$= y\left(x^2+xy+\frac{y^2}{4}\right),$$

$$\therefore 2+y\sqrt{x} = \pm\sqrt{y}\left(x+\frac{y}{2}\right) \ldots\ldots\ldots (1)$$

This equation splits into two others. Taking +

$$\left.\begin{array}{l} y\sqrt{y}-2y\sqrt{x}+2x\sqrt{y}-4 = 0 \ldots\ldots\ldots (2)\\ y\sqrt{y}+2y\sqrt{x}+2x\sqrt{y}+4 = 0 \ldots\ldots\ldots (3)\end{array}\right\}$$

The second of the given equations gives

$$x\sqrt{y}-y\sqrt{x} = x\sqrt{x}-3 \ldots\ldots\ldots (4)$$

Adding (4) to (2) we get

$$y^{\frac{3}{2}}-3yx^{\frac{1}{2}}+3xy^{\frac{1}{2}}-1=x^{\frac{3}{2}}$$

$$\therefore\ y^{\frac{1}{2}}-1=x^{\frac{1}{2}} \ldots\ldots\ldots (5)$$

Substituting $y^{\frac{1}{2}}=1+x^{\frac{1}{2}}$ in (4), we get

$$x+x\sqrt{x}-(1+x^{\frac{1}{2}})^2.\sqrt{x} = x\sqrt{x}-3,$$

$$\therefore\ x^{\frac{3}{2}}+x^{\frac{2}{2}}+x^{\frac{1}{2}} = 3,$$

$$\text{and } -4x^{\frac{2}{2}}+2x^{\frac{1}{2}}-1 = -4x+2x^{\frac{1}{2}}-1,$$

$\therefore$ adding these equals,

$$(x^{\frac{1}{2}}-1)^3 = 2(1-2x+x^{\frac{1}{2}}) = -2(\sqrt{x}-1)(1+2\sqrt{x})$$

$$\therefore\ (\sqrt{x}-1)(x-2\sqrt{x}+1+2+4\sqrt{x}) = 0,$$

$$\therefore\ (\sqrt{x}-1)(x+2\sqrt{x}+3) = 0,$$

$$\left.\begin{array}{l}\therefore\ x=1\\ \text{and } x+2\sqrt{x}+3=0\end{array}\right\} \ldots\ldots\ldots \begin{array}{r}(6)\\(7)\end{array}$$

From (6) and (5)

$$y=4 \ldots\ldots\ldots (8)$$

Also from (7) $\sqrt{x} = -1\pm\sqrt{-2}$,

$$\therefore\ x = -1\mp 2\sqrt{-2} \ldots\ldots\ldots (9)$$

And from (9) and (5),

$$\sqrt{y} = 1+\sqrt{x} = \pm\sqrt{-2},$$

$$\therefore\ y = -2 \text{ . . . . } (10)$$

$\therefore$ simultaneous roots are

$$\left.\begin{array}{l} x = 1 \\ y = 4 \end{array}\right\};\ \left.\begin{array}{l} x = -1-2\sqrt{-2} \\ y = -2 \end{array}\right\};\ \left.\begin{array}{l} x = -1+2\sqrt{-2} \\ y = -2 \end{array}\right\}.$$

By combining (3) with (4) other sets of simultaneous roots may be found.

5. Let $x =$ A's rate per minute,

$y =$ B's . . . . .

$$\therefore\ 6x + 2 = \text{length of course,}$$

$$= 5y + y + 20,$$

$$\text{Also, } 4(x-y) = \frac{6x+2}{440},$$

$$\left.\begin{array}{r} \therefore\ 6x - 6y = 18 \\ 880y - 874x + 2 = 0 \end{array}\right\},$$

$$\left.\begin{array}{r} \therefore\ y = x-3, \\ \therefore\ 880(x-3) - 874x+2 = 0 \end{array}\right\},$$

$$\therefore\ 6x + 2 = 2640 \text{ yards} = \text{length of course.}$$

6. Let $x =$ original revenue,

$y =$ interest of debt,

$z =$ expense of collecting x,

$\therefore\ \frac{9}{4}x =$ increased income,

$\frac{3}{2}z =$ expense,

$$\left.\begin{array}{l} \therefore\ \frac{9}{4}x - y - \frac{3}{2}z : x - y - z :: \frac{5\frac{1}{3}}{1\frac{1}{3}} : 1 \\ \text{and } \frac{9}{16}x - y - \frac{3}{4}z : x - y - z :: \frac{3}{2\frac{3}{3}} : 1 \\ \text{and } \frac{9}{16}x - y - \frac{3}{4}z = 4000000 \end{array}\right\},$$

$$\left.\begin{array}{r} \therefore\ 117x - 52y - 78z = 204(x - y - z) \\ 207x - 368y - 276z = 48(x - y - z) \\ 9x - 16y - 12z = 64000000 \end{array}\right\},$$

$$\left.\begin{array}{rl} \therefore\ 87x - 152y - 126z = 0 & \text{. . . . . . } (1) \\ 159x - 320y - 228z = 0 & \text{. . . . . . } (2) \\ 9x - 16y - 12z = 64000000 & \text{. . } (3) \end{array}\right\}.$$

Multiplying (1) by 2, and (3) by 21, and subtracting, we have

$$15x - 32y = 1344000000.$$

Multiplying (3) by 19, and subtracting (2), we have

$$12x + 16y = 1216000000,$$
$$\therefore\ 3x + 4y = 304000000,$$
$$\left.\begin{array}{r}\text{and } 15x + 20y = 1520000000\\ \text{But } 15x - 32y = 1344000000\end{array}\right\},$$
$$\therefore\ 52y = 176000000,$$
$$\left.\begin{array}{r}\text{Also } 15x - 32y = 1344000000\\ 24x + 32y = 2432000000\end{array}\right\},$$
$$\therefore\ 39x = 3776000000,$$
$$\therefore\ x = 96820512\tfrac{32}{39}l.$$
$$y = \frac{44000000}{13} = 3384615\frac{5}{13}l.$$

z may hence be found.

7. Let x yards per minute be A's velocity at starting,

y . . . . B's . . . .

z . . . . C's . . . .

w yards the length of A's stroke,

v that of B's.

Then, since velocity $\propto$ number of strokes $\times$ length of them, *dato tempore*,

$$\therefore\ x : z :: w : w-1 \quad . \quad . \quad . \quad . \quad (1).$$

Again, whilst A takes 42 strokes, B takes $\frac{7}{6} \times 42$ or 49, and since A has then got $42w + 20$ yards from B's starting place, by the question, we have

$$v + (v+\tfrac{1}{12})+(v+\tfrac{2}{12}) + \ldots v + \tfrac{48}{12} = 42w + 20 - 1 + 16,$$
$$\text{or } (2v + 4)\tfrac{49}{2} = 42w + 35,$$
$$\text{or } 70 = 6w - 9\ .\ .\ (2).$$

Again, A having now gone $42w$ yards; C, $42w-42$ yards; and B, $42w - 1$ yards; it follows that C is 61 yards behind B. Also now C's strokes are exactly as quick as B's. Hence, by the question,

$$v + \tfrac{48}{12} + (v + \tfrac{46}{12}) + (v + \tfrac{44}{12}) + \ldots .\ 28 \text{ terms} = w-1$$
$$+ (w-1+\tfrac{1}{6}) + (w-1+\tfrac{2}{6}) + \ldots . \text{ to } 28 \text{ terms} + 11 - 61,$$
$$\text{or } (2v + 8 - \tfrac{1}{6} \times 27)\,14 = (2w - 2 + \tfrac{1}{6} \times 27)\,14 - 50,$$

which gives

$$7v = 7w - 16\ .\ .\ .\ .\ .\ .\ .\ .\ (3).$$

From (2) and (3)

$$w = 7,\ v = 4\tfrac{5}{7} \text{ yards per stroke.}$$

Finally, from (1)

$$x : z :: 7 : 6 \ldots\ldots\ldots (4).$$

$$x : y :: w : v \times \tfrac{7}{6} :: 42 : 33 :: 14 : 11.$$

So that the velocities of A, B, C are respectively as

14, 11, 12.

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1. Ans. $1\frac{3}{5}$,

2. Ans. $13\frac{1}{7}$,

3. Ans. In 104 days.

4. The Danish stock is worth $2350 \times \dfrac{75\frac{1}{4}}{100}$, or $\dfrac{47 \times 30}{8}l$.

This will buy Russian stock =

$$\frac{100}{110\frac{7}{8}} \times \frac{47 \times 301}{8} = \frac{100 \times 47 \times 301}{887},$$

$$\text{or } \frac{1414700}{887}.$$

Now, interest of Danish stock

$$= \frac{3}{100} \times 2350 = \frac{3 \times 47}{2} = \frac{141}{2}l.$$

and that of the Russian

$$= \frac{1}{20} \times \frac{1414700}{887} = \frac{70735}{887} = 79\frac{662}{887}l.$$

$\therefore$ difference required

$$= 9\frac{662}{887} - \frac{1}{2},$$

$$= 9\frac{437}{1774}l.$$

5. The G. C. M. is $x^{-2} + 2x^{-1} + 1$, or $\left(\dfrac{1+x}{x}\right)^2$.

6. Each set of three consonants and one vowel may make 4. 3. 2. 1 words; hence, with three *given* consonants, and the five vowels, there may be made 5. 4. 3. 2. 1 words.

Again, there are $\frac{19\times 18\times 17}{1.\ 2.\ 3}$ *different* sets of 3 consonants.

Consequently, the number of words required is

$$\frac{19\times 18\times 17\times 4\times 3\times 2\times 1}{1\times 2\times 3} \text{ or } 23256.$$

7. See *Private Tutor*, vol. i, pp. 22, 262.

8. $(27^{\frac{1}{2}}-25^{\frac{1}{3}})(27^{\frac{1}{2}}+25^{\frac{1}{3}}) = 27-25^{\frac{2}{3}} = 27-625^{\frac{1}{3}}$,

$$= -\left\{\left(\frac{625}{27^3}\right)^{\frac{1}{3}} - 1\right\}\times 27.$$

But $x^3-1 = (x-1)(x^2+x+1)$,

$$\therefore \left\{\left(\frac{625}{27^3}\right)^{\frac{1}{3}} \quad 1\right\}\left\{\left(\frac{625}{27^3}\right)^{\frac{2}{3}}+\left(\frac{625}{27^3}\right)^{\frac{1}{3}}+1\right\} = \left(\frac{625}{27^3}-1\right).$$

$$\therefore \text{ if } n = (27^{\frac{1}{2}}+25^{\frac{1}{3}})\left\{\left(\frac{625}{27^3}\right)^{\frac{2}{3}}+\left(\frac{625}{27^3}\right)^{\frac{1}{3}}+1\right\},$$

$$= (27^{\frac{1}{2}}+25^{\frac{1}{3}})\left(\frac{25\times 25^{\frac{1}{3}}}{27^2}+\frac{5\times 5^{\frac{1}{3}}}{27}+1\right),$$

the expression will be rational, being $= 27-\frac{625}{27^2} = \frac{19058}{729}$.

9. (1). Gives ans. $\frac{1-x}{\sqrt[3]{x}}$.

(2). Ans. 2. $\frac{y^4-x^4+2x^2y^2}{x^4-y^4}$.

(3). Extracting the root, it is $\pm(1-\frac{3}{4}x^{\frac{1}{3}}+x^{\frac{1}{2}})$.

10. The first quotient is $\frac{5a^4}{b^3}-b^2+3b^3$.

The second is $x^{p(q-1)}+x^{p(q-2)}+\ldots\ x^0$,

$\therefore$ the last term is 1.

11. $\sqrt[3]{(-4-10\sqrt{-2})} = 2-\sqrt{-2}$.

Also, $\sqrt[4]{\left(\frac{17}{3} - 4\sqrt{2}\right)} = \frac{1}{3}(\sqrt{6}-\sqrt{3}).$

12. Let x be the common difference of the reciprocals of the terms; then

$$\tfrac{1}{2} + \frac{1}{2+x} + \frac{1}{2+2x} = \frac{11}{12}.$$

$$\text{Hence, } \frac{2+2x+2+x}{2(x^2+3x+2)} = \frac{5}{12},$$

which gives $x = 2$ and $-\frac{7}{5}$

and the series continued both ways

$$\ldots\ldots -\tfrac{1}{4},\ -\tfrac{1}{2},\ \infty,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{6},\ \tfrac{1}{8}\cdot\ldots\ldots$$

and $\ldots\ldots \frac{5}{31},\ \frac{5}{24},\ \frac{5}{17},\ \frac{1}{2},\ \frac{5}{3},\ -\frac{5}{4},\ -\frac{5}{11}\cdot\ldots\ldots$

13. Let r be the common ratio of the terms, when n Geometric means are inserted between x and u; then

$$x\,r^{n+2-1} = u.$$

$$\therefore\ r = \left(\frac{u}{x}\right)^{\frac{1}{n+1}},$$

$$\text{and the } n^{\text{th}} \text{ mean is } x\,r^n = x\left(\frac{u}{x}\right)^{\frac{n}{n+1}},$$

Hence,

$$M_1 = x.\left(\frac{y}{x}\right)^{\frac{n}{n+1}},\ M_2 = 2^{\frac{n}{n+1}}\,x\left(\frac{y}{x}\right)^{\frac{n}{n+1}},\ M_3 = 4^{\frac{n}{n+1}}\,x\left(\frac{y}{x}\right)^{\frac{n}{n+1}} +$$

&c.

$$\therefore\ \frac{M_1}{M_2} = \frac{1}{2^{\frac{n}{n+1}}},\ \frac{M_2}{M_3} = \left(\frac{1}{2}\right)^{\frac{n}{n+1}},\ \frac{M_3}{M_4} = \left(\frac{1}{2}\right)^{\frac{n}{n+1}} \text{ to } n \text{ terms},$$

$$\therefore\ \text{their sum} = n.\left(\frac{1}{2}\right)^{\frac{n}{n+1}} = \left(\frac{n^{n+1}}{2^n}\right)^{\frac{1}{n+1}}.$$

14. Let C and C′ be the combinations taken r together and $(r+1)$ together respectively; then

$$C = \frac{m(m-1)\ldots(m-r+2)}{1.\ 2.\ldots\ldots(r-1)},\ C' = \frac{m(m-1)\ldots.(m-r+1)}{1.\ 2.\ldots r},$$

$$\therefore\ \frac{C'}{C} = \frac{m-r+1}{r} = \frac{m+1}{r} - 1.$$

But, if for any one value of r, C′ be < C; then, for all higher values of r, C′ is still *a fortiori* less than C,

$\therefore$ when the number of combinations is greatest,

$$\frac{m+1}{r}-1 \text{ is } = \text{ or } < 1,$$

$$\text{and } \therefore r \text{ is } = \text{ or } > \frac{m+1}{2}.$$

15. First, $a^2+b^2-a^2 = \dfrac{(a^2+b^2-c^2-d^2)\,ab}{ab-cd}$,

$$\therefore 1-\left(\frac{a^2+b^2-a^2}{2ab}\right)^2 = 1-\frac{(a^2+b^2-c^2-d^2)^2}{4\,(ab-cd)^2},$$

$$=\frac{4\,(ab-cd)^2-(a^2+b^2-c^2-d^2)^2}{4\,(ab-cd)^2},$$

$$=\frac{\{2(ab-2cd)+a^2+b^2-c^2-d^2\}\,(2ab-2cd-a^2-b^2+c^2+d^2)}{4\,(ab-cd)^2},$$

$$=\frac{\{(a+b)^2-(c+d)^2\}\,\{(c-d)^2-(a-b)^2\}}{4\,(ab-cd)^2},$$

$$=\frac{(a+b+c+d)\,(a+b-c-d)\,(a-b+c-d)\,(b+c-a-d)}{4\,(ab-cd)^2}.$$

QUEEN'S COLLEGE, 1820.

1. This is the best method in practice;

$$\begin{aligned} \text{Let } S &= .1293131\ \ldots\ldots \\ \text{then } 100000\ S &= 12931.31\ \ldots\ldots \\ \text{and } 1000\ S &= 129.\ 31\ \ldots\ldots \\ \therefore\ 99000\ S &= 12802 \\ \therefore S &= \frac{12802}{99000} = \frac{1067}{8250}. \end{aligned}$$

2. The G. C. M. is $x-2$.

3. The root is .00135.

4. At simple interest, at five per cent. (*Wood*, art. 392.)

$$P = \frac{M}{1+nr},$$

$$\therefore \text{ present value required } = \frac{90}{1+\frac{1}{80}} + \frac{735}{1+\frac{7}{240}} + \frac{450}{1+\frac{11}{240}},$$

$$= \frac{800}{9} + \frac{240 \times 735}{247} + \frac{45 \times 2400}{251} l.$$

which is easily reduced.

5. Number $= a+b.\ 10+c.\ 10^2+ \ldots\ldots$

$= a+b.\ (3+7)+c\,(3+7)^2+ \ldots\ldots$

$= a+b.\ 7+c.\ 7^2+d.\ 7^3+ \ldots + \text{quantity} \times 3.$

$\therefore$ &c.

6. $\sqrt{n}$ is $> \sqrt[3]{(n+1)}$ if n^3 is $>(n+1)^2$.

Let $n=1+m$, in which m may be any quantity, however small or great; then

$$\sqrt{n} \text{ is } > \sqrt[3]{(n+1)} \text{ if } (1+m)^3 \text{ be } >(2+m)^2,$$

whence it is evident the proposition is not true; for, even when $m=1$, $(1+1)^3$ is $<(2+1)^2$.

7. For

$$(a+k)^3-(b+k)^3=a^3-b^3+3\,(a^2-b^2)\,k+3\,(a-b)\,k^2.$$

8. $(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+$ &c.

9. (1). Sum $= \dfrac{3^n-(-4)^n}{14 \times 3^n}$.

(2). Sum $= \dfrac{n\,(3n+1)}{4}$.

(3). Here is an error of the press. The series should be

$$\frac{1}{1+\sqrt{x}} + \frac{1}{(1-x)(1+\sqrt{x})} + \frac{1}{(1-x)^2(1+\sqrt{x})} + \text{\&c.}$$

$$\text{or } \frac{1}{1+\sqrt{x}}\left(1 + \frac{1}{1-x} + \frac{1}{(1-x)^2} + \text{\&c.}\right).$$

Sum of n terms =

$$\frac{1}{1+\sqrt{x}} \cdot \frac{\left(\frac{1}{1-x}\right)^n - 1}{\frac{1}{1-x} - 1} = \frac{1-(1-x)^n}{x(1+\sqrt{x})(1-x)^{n-1}}.$$

The sum to $\infty = \infty$.

(4).

$$\left.\begin{aligned} S &= \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \ldots\ldots \infty \\ &+ \frac{1}{r^2} + \frac{2}{r^3} + \ldots\ldots \infty \end{aligned}\right\} = \frac{1}{r} \cdot \frac{1}{1-\frac{1}{r}} + \frac{1}{r} \cdot S,$$

whence $S = \dfrac{r}{(r-1)^2}$.

10.

$$\text{No.} = n + \frac{n(n-1)}{1.\ 2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} + \ldots \frac{n(n-1)\ldots 3.2.1}{1.\ 2\ldots(n-1)n}$$

But $(1+1)^n = 1 + n + \dfrac{n(n-1)}{1.\ 2} + \ldots . 1,$

$\therefore$ No. $= 2^n - 1$.

11. (1). $x = \frac{1}{2}$.

(2). $x = \frac{2}{3}$ and -3.

(3). $x = 2$.

(4). $x = \dfrac{2}{m \pm \sqrt{(2n-m^2)}}$,

$y = \dfrac{2}{m \mp \sqrt{(2n-m^2)}}$.

(5). $x = 9$ and 4.

(6). $x^3 + 1 = p.(x+1)$,

$\therefore (x+1)(x^2 - x + 1 - p) = 0$,

$\therefore x = -1$, and $\dfrac{1 \pm \sqrt{(4p-3)}}{2}$.

(7). $x^3 + 1 + px(x+1) = 0$,

$\therefore (x+1)(x^2 - x + 1 + px) = 0$,

$\therefore x = -1$, and $\dfrac{1 - p \pm \sqrt{(p^2 - 2p - 3)}}{2}$.

12. Make $y = \frac{1}{x}$; then (*Wood*, art. 381.),

$$\text{and } y^3 + \frac{q}{r}\,y^2 + \frac{p}{r}\,y + \frac{1}{r} = 0,$$

whose roots are in Arithmetic Progression. (See p. 84, of this work.)

13. Let x and y be the digits; then

$$\begin{aligned} x^2 + y^2 &= 25, \\ xy &= 12, \\ \therefore\ x + y &= \pm 7, \\ x - y &= \pm 1, \\ \therefore\ x = 4,\ y &= 3, \end{aligned}$$

$\therefore$ the number is 34 or 43.

14. Let x be the number of moidores, y that of guineas; then

$$27\,x - 21\,y = 12 \times 20,$$

$$\left.\begin{aligned} \text{or } 9\,x - 7\,y &= 80 \\ \text{and } x + y &= 16 \end{aligned}\right\} \therefore\ x = 12,\ y = 4.$$

15.

$\sqrt{(-4+2\sqrt{-5})}, = 1 + \sqrt{-5}$, and $\sqrt{(2\sqrt{-1})} = 1 + \sqrt{-1}$.

16. Let a be the first term, l the last term, and s the sum. Also x the number of terms, and y the common ratio. Then, since

$$\left.\begin{aligned} s &= a.\,\frac{y^x - 1}{y - 1} \\ \text{and } l &= ay^{x-1} \end{aligned}\right\},$$

$$\therefore\ ay^x = ly,$$

$$\therefore\ s = \frac{ly - a}{y - 1},$$

$$\therefore\ y = \frac{a - s}{l - s}, \text{ and } x = \frac{\log l - \log a}{\log y} + 1,$$

$$= \frac{\log l - \log a}{\log (a - s) - \log (l - s)} + 1.$$

QUEEN'S COLLEGE, 1826.

1. Ans. 0.8332 . . . . . moidores.

2. $\sqrt{(379.864)} = 19.\ 4901$ . . . . .

and $\sqrt[3]{(.00\dot{0}05\dot{7}46\dot{4}0)} = 0.\ 038$ . . . . .

3. Every odd square number is of the form $4n+1$.
Let $m = 4n+1$; then

$$\frac{(m+3)(m+7)}{32} = \frac{(4n+4)(4n+8)}{32} = \frac{(n+1)(n+2)}{2},$$

one of whose factors must be divisible by 2, and $\therefore$ &c.

4. n may $= 10^{q-1}$, but cannot $= 10^{q}$,

m . . . $= 10^{p-1}$, ... $= 10^{p}$,

$\therefore \dfrac{n}{m}$ . . . $= 10^{q-p}$, or may have $q-p+1$ digits,

&c. &c.

This is also proved reversely from the like rule in multiplication.

5. $x-a$ is the G. C. M. required, and the reduced fraction is

$$\frac{3x-a-2b}{(x-a)(x-b)}.$$

6. Ans. $x = 12$ for the first equation,

$x = 8$ for the second,

$x = 5$ and $-4\frac{3}{7}$ for the third,

$x = 4$ and $-\frac{191}{67}$ for the fourth,

$x = 16$ for the fifth.

$\left.\begin{array}{l} x = 1 \\ y = -1 \end{array}\right\}$ for the sixth,

For the seventh, make $y = ux$, and we easily get

$$u = \frac{a^2+b^2 \pm \sqrt{\{4a^2 b^2 + (a^2+b^2)^2\}}}{2a^2},$$

Whence x and y are easily obtained.

7. Let x and y be the numbers; then

$$\left.\begin{aligned} x^2 + y^2 &= 3xy \\ \text{and} \quad x^2 - y^2 &= \frac{x}{y} \end{aligned}\right\}$$

By subtraction, $2y^2 = x\left(3y - \frac{1}{y}\right)$, $\therefore\ 2y^3 = x(3y^2 - 1)$,

$$\therefore\ x = \frac{2y^3}{3y^2-1},$$

and subtracting in the first

$$\frac{4y^6}{(3y^2-1)^2} + y^2 = \frac{6y^4}{3y^2-1},$$

$$\therefore\ 4y^4 + 9y^4 - 6y^2 + 1 = 6y^2(3y^2-1) = 18y^4 - 6y^2$$

$$\therefore\ 5y^4 = 1,$$

$$\therefore\ y = (\tfrac{1}{5})^{\frac{1}{4}} \text{ whence } x = \tfrac{1}{4}\cdot(\tfrac{1}{5})^{\frac{1}{4}}(3 + \sqrt{5}).$$

8. The Harmonic Mean between x and z is $\dfrac{2xz}{x+z}$,

But $x = \dfrac{a+b}{2}$ and $z = \sqrt{(ab)}$,

$$\therefore\ y = 2.\left(\frac{a+b}{2}.\ \sqrt{ab}\right) \div \left(\frac{a+b}{2} + \sqrt{ab}\right),$$

$$= 2.\ \frac{(a+b)\sqrt{ab}}{(\sqrt{a}+\sqrt{b})^2} = 2.\ \frac{a+b}{\left\{\sqrt{\frac{a}{\sqrt{(ab)}}} + \sqrt{\frac{b}{\sqrt{(ab)}}}\right\}^2};$$

$$= 2.\ \frac{a+b}{\left\{\sqrt[4]{\frac{a}{b}} + \sqrt[4]{\frac{b}{a}}\right\}^2}.$$

9. (1). Common difference $= \frac{1}{2}$,

Sum $= (2 + \frac{11}{2})\ 6 = 45$,

(2). Common ratio $= \frac{3}{2}$,

$$\text{Sum} = \frac{(\frac{3}{2})^{10} - 1}{\frac{3}{2} - 1} = \frac{3^{10} - 2^{10}}{2^9} = \frac{58025}{512}.$$

(3). Common ratio $= -\frac{2}{5}$,

$$\text{Sum} = \frac{1}{1 + \frac{2}{5}} = \frac{5}{7}.$$

(4). The general term $= (2n + 1)(2n + 3)$. (See *Wood*, art. 427.)

$$\text{Sum} = \frac{(2n-1)(2n+1)(2n+3)}{6} + \tfrac{1}{2}.$$

10. These are both recurring series, whose scales of relation (*Wood*, art. 414.) are

$$f + g = 2 - 1, \text{ and } f + g = -2 - 1,$$

$$\therefore\ S = \frac{1+2x-2x}{1-2x+x^2} = \frac{1}{(1-x^2)}, \text{ and } S' = \frac{1-2x+2x}{1+2x+x^2} = \frac{1}{(1+x)^2},$$

11. $(1-x^2)^{\frac{7}{3}} = 1 - \frac{7}{3}x^2 + \frac{14}{9}x^4 - \frac{14}{81}x^6 - \frac{7}{243}x^8 + \&c.$

12. $\text{It} = a\left(1 - n + n\,\frac{n-1}{2} - \&c.\right)$

$$- b\left(n - 2n.\frac{n-1}{2} + 3.\,\frac{n.(n-1)(n-2)}{1.\,2.\,3} - \&c.\right.$$

$$\text{But } (1-1)^n = 1 - n + n.\frac{n-1}{2} - \&c. = 0,$$

$$\therefore\ \text{It} = -b\left(n - 2n.\frac{n-1}{2} + 3.\,\frac{n(n-1)(n-2)}{1.\,2.\,3} - \&c.\right)$$

$$= -bn\left\{1-(n-1)+\frac{(n-1)(n-2)}{2} - \frac{(n-1)(n-2)(n-3)}{2.\,3} + \&c.\right.$$

$$= -bn\,(1-1)^{n-1} = 0.$$

13. $\frac{a^x}{(a+b)^x} = \left(1 + \frac{b}{a}\right)^{-x} = 1 - \frac{b}{a}\,x$ nearly.

14. $\sqrt{(45 + 28\sqrt{-1})} = 7 + 2\sqrt{-1}.$

and $\sqrt[3]{(16+8\sqrt{5})} = 2\sqrt[3]{(2+\sqrt{5})} = 2.\,\frac{\sqrt{5}+1}{2} = \sqrt{5}+1.$

15. (1). Let α, $-\alpha$, β, be the roots; then

$$\alpha - \alpha + \beta = -4, \text{ or } \beta = 4,$$

Hence, the roots are 3, -3, and 4.

(2). Let $a-b$, a, $a+b$, be the roots; then

$$a-b+a+a+b = -9,$$

$$\therefore\ a = -3,$$

$$\text{and } (a-b)(a+b)\,a = -15,$$

$$\therefore\ 9 - b^2 = 5,$$

$$b = \pm 2,$$

$\therefore$ the roots are $-5, -3, -1$.

(3). Let a, b, c, be the roots; then

$$\left.\begin{aligned} a+b+c &= 7 \\ \text{and } a+b &= 8 \end{aligned}\right\},$$

$$\therefore\ c = -1,$$

$$\text{and } abc = -15,$$

$$\therefore\ ab = 15,$$

$$a^2+2ab+b^2 = 64,$$

$$4ab = 60,$$

$$\left.\begin{aligned} \therefore\ a-b &= \pm 2 \\ a+b &= 8 \end{aligned}\right\},$$

$$\therefore\ a = 5,\ b = 3,$$

and the roots are $5, 3, -1$.

16. Let the origin of co-ordinates be at the extremity of the base, the axis of x being along the base; and let base $= a$, and given ratio $= b$; then, if s be one side,

$$y^2 = s^2-x^2 = b^2s^2-(a-x)^2,$$

$$\therefore\ s^2 = \frac{(a-x)^2-x^2}{b^2-1},$$

$$\therefore\ y^2 = \frac{(a-x)^2-x^2}{b^2-1}-x^2 = \frac{a^2-2ax-(b^2-1)x^2}{b^2-1},$$

$\therefore$ the locus is a conic section, which it is easy to describe. See *Wood*, Alg. pt. iv.

QUEEN'S COLLEGE, 1827.

1. $\sqrt{.008281} = .091$,

and $\sqrt[3]{.000405224} = .074$.

2. Let x be the number of days required; then, since for the same work,

$$\text{number of days} \propto \frac{1}{\text{number of men}},$$

$$x : m :: \frac{1}{a+b} : \frac{1}{a},$$

$$\therefore x = \frac{ma}{a+b}.$$

There appears to be a superfluous condition in the enunciation.

3. The value required $= \dfrac{82\frac{7}{8}}{100} \times 1743l.$ sterling.

$$= 1444l.\ 10s.\ 2\tfrac{1}{16}d.$$

4. See p. 49.

5. At simple interest, (*Wood*, art. 391.),

$$M = P(1 + nr).$$

At compound interest, (*Wood*, 397.),

$$M' = P\,R^n;$$

$\therefore$ by the question,

$$P(1+2r) = \tfrac{7}{8} P. (1+r)^2,$$

$$\therefore 8 + 16r = 7 + 14r + 7r^2,$$

$$\text{which gives } r = \frac{1 + 2\sqrt{2}}{7} = \frac{3.82}{7} \text{ nearly,}$$

$$= \frac{54\frac{4}{7}}{100}.$$

6. Every odd square number is of the form $4n+1$, (*Barlow's* Theory of Numbers); but any number, whose two last digits are 9, is the form $9 + 9 \times 10 + 100 \times N$;

if this be a square number, and $\therefore$ of the form $4n+1$,

$$8 + 9 \times 10 + 100 \times N,$$

it is divisible by 4, and $\therefore 9 \times 10$ is divisible by 4; which it is not. $\therefore$ "No number whose," &c.

7. See *Barlow's* Theory of Numbers.

8. The G. C. M. $= x - 7$,

$$\text{and the reduced fraction} = \frac{x^2 + a^2}{x + b}.$$

9. First, $uv = \frac{1}{4}\left(x + \dfrac{1}{x}\right)\left(y + \dfrac{1}{y}\right)$,

$$= \tfrac{1}{4}\left(xy + \frac{y}{x} + \frac{x}{y} + \frac{1}{xy}\right),$$

Also, $1-u^2 = 1 - \frac{1}{4}\left(x + \frac{1}{x}\right)^2 = -\frac{1}{4}\left(x - \frac{1}{x}\right)^2$,

and $1-v^2 = 1 - \frac{1}{4}\left(y + \frac{1}{y}\right)^2 = -\frac{1}{4}\left(y - \frac{1}{y}\right)^2$,

$$\therefore \sqrt{(1-u^2)}.\sqrt{(1-v^2)} = \frac{1}{4}\left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right),$$

$$= \frac{1}{4}\left(xy - \frac{y}{x} - \frac{x}{y} + \frac{1}{xy}\right),$$

$$\therefore uv + \sqrt{(1-u^2)}\sqrt{(1-v^2)} = \frac{1}{2}\left(xy + \frac{1}{xy}\right).$$

10. The index being n, the p^{th} term being P, and $(p+2)^{\text{th}}$ $=$ Q, we have

$$\frac{n.(n-1)(n-2)\ldots\ldots(n-p+2)}{1.\ 2.\ 3\ldots\ldots(p-1)}a^{p-1} = P,$$

$$\text{and } \frac{n(n-1)(n-2)\ldots\ldots(n-p+4)}{1.\ 2.\ 3\ \ldots\ldots(p-3)}a^{p-3} = Q,$$

$$\therefore \frac{(n-p+3)(n-p+2)}{(p-2)(p-1)}.a^2 = \frac{P}{Q},$$

whence n is easily found from the quadratic.

11. See *Private Tutor*, vol. i. p. 263.

12. Let x be the first term, and y the required ratio; then a being the given sum to ∞, and b the sum of the cubes, we have

$$a = \frac{x}{1-y},$$

$$\text{and } b = x^3 + x^3y^3 + x^3y^6 + x^3y^9 + \text{ \&c. to } \infty,$$

$$= x^3(1 + y^3 + y^6 + y^9 + \ldots\ldots\infty),$$

$$= x^3.\frac{1}{1-y^3},$$

$$= a^3.\frac{(1-y)^3}{1-y^3} = a^3.\frac{(1-y)^2}{1+y+y^2},$$

$$\therefore by^2 + by + b = a^3 - 2a^3y + a^3y^2,$$

$$\therefore y^2 - \frac{2a^3 - b}{a^3 - b}y = -1,$$

whence y, and then x, are easily found.

13. Let x be the common difference of the reciprocals of the series, which are in Arithmetic Progression; then these reciprocals are

$$2,\ 2+x,\ 2+2x,$$

and, by the question,

$$\frac{1}{2}+\frac{1}{2+x}+\frac{1}{2+2x}=1\frac{1}{12}=\frac{13}{12},$$

$$\therefore\ (4+3x)\,12=7\,(4+6x+2x^2),$$

which gives $x=1$, or $-\frac{10}{7}$,

$\therefore$ the series is either

$$\ldots\ldots -1,\ \infty,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \&\text{c.}$$

$$\text{or, } \ldots\ldots\ldots\ \tfrac{7}{24},\ \tfrac{1}{2},\ \tfrac{7}{4},-\tfrac{7}{6}.$$

14. (1). The Common Difference $=-\frac{1}{7}$, and Sum $=0$.

(2). Sum $=\dfrac{105469}{354294}$.

(3). Common Ratio $=\dfrac{1}{2\sqrt{-1}}$,

$$\text{and Sum}=\frac{\frac{1}{3}}{1-\dfrac{1}{2\sqrt{-1}}}=\frac{4+2\sqrt{-1}}{15}.$$

15. (1). Ans. $x=16$.

(2). $x=6$, and $-\frac{445}{47}$,

(3). $x=7$, and $-\frac{58}{7}$.

(4). First $x^2+9+\sqrt{(x^2+9)}+\frac{1}{4}=x^4-x^2+\frac{1}{4}$,

$$\therefore\ \sqrt{(x^2+9)}+\tfrac{1}{2}=x^2-\tfrac{1}{2},$$

$$\therefore\ x^2+9=(x^2-1)^2,$$

$$=x^4-2x^2+1,$$

Hence, $x=\pm\frac{1}{2}\sqrt{(6\pm2\sqrt{41})}$.

(5). Cubing the second,

$$x-y+3\sqrt[3]{(xy)}.\ (\sqrt[3]{x}-\sqrt[3]{y})=b,$$

$$\therefore\ \sqrt[3]{(xy)}=\frac{b-a}{3\sqrt[3]{b}},$$

$$\therefore\ xy=\frac{(b-a)^3}{27b}.$$

Hence $x+y = \pm\sqrt{\left\{a^2 + \frac{4(b-a)^3}{27b}\right\}}$,

but $x-y = a$.

Whence x and y are easily found.

(6). These being homogeneous, make $y = xu$; then $u^2 + \frac{61}{6}u = \frac{55}{6}$ will give u; whence x and y.

16. $\sqrt{(21+\sqrt{-400})} = 5+2\sqrt{-1}$,

$$\sqrt[5]{(76+44\sqrt{3})} = 1+\sqrt{3}.$$

17. $m\,(3^{\frac{2}{3}} + 5^{\frac{1}{2}}) = m.\ \{(3^4)^{\frac{1}{6}} + (5^3)^{\frac{1}{6}}\}$,

and $\therefore\ m = (3^4)^{\frac{5}{6}} - (3^4)^{\frac{4}{6}}.(5^3)^{\frac{1}{6}} + (3^4)^{\frac{3}{6}}(5^3)^{\frac{2}{6}} - (3^4)^{\frac{2}{6}}(5^3)^{\frac{3}{6}}$

$$+ (3^4)^{\frac{1}{6}}(5^3)^{\frac{4}{6}} - (5^3)^{\frac{5}{6}},$$

$$= 3^{\frac{10}{3}} - 3^{\frac{8}{3}}.5^{\frac{1}{2}} + 3^2.5 - 3^{\frac{4}{3}}.5^{\frac{3}{2}} + 3^{\frac{2}{3}}.5^2 - 5^{\frac{5}{2}},$$

$$= 27\sqrt[3]{3} - 9\sqrt[3]{9} + 45 - 3\sqrt[3]{3}\times 5\sqrt{5} + 25\sqrt[3]{9} - 25\sqrt{5},$$

$$= 45 + 27\sqrt[3]{3} + 16\sqrt[3]{9} - 25\sqrt{5} - 15.\sqrt{5}.\sqrt[3]{3}.$$

18. See the Preface to any System of Logarithms.

The same series, although it may be the best for a certain set of numbers, is not the best generally.

19. Since $\sqrt{19} = 4+\sqrt{19}-4 = 4 + \frac{3}{\sqrt{19}+4}$,

$$\therefore\ \sqrt{19} = 4 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{3}{3+1,\ \&c.}}}$$

20. (1). By the Method of Divisors, its roots are found to be 3, 4, and $-$ 1.

(2). The equations

$$\left.\begin{array}{l} x^4-5x^3-5x^2+45x-36 = 0 \\ x^4+5x^3-5x^2-45x-36 = 0 \end{array}\right\},$$

have a common measure of the form x^2-a^2,

and $\therefore$ their difference is divisible by a quantity of the form

$x^2 - a^2$. Consequently $x^2 - 9$ is that factor, and two of the roots are ± 3.

(3). This is a misprint for $a^x \times b^{-x} = c$.

$$\text{Ans. } x = \frac{\log c}{\log a - \log b}.$$

21. (1) The $(n+1)^{\text{th}}$ term of $3.5 + 5.7 + 7.9 + \ldots$ is

$$(2n+3)(2n+5).$$

$\therefore$ *Wood's* Alg. art. 427.

$$3.5 + 5.7 + 7.9 + \ldots n \text{ terms} = \frac{2n+1)(2n+3)(2n+5)}{6} - \tfrac{5}{2},$$

$$\therefore \text{ sum required} = \frac{4n^3 + 18n^2 + 23n - 30}{3}.$$

(2). The $(n+1)^{\text{th}}$ term is $\dfrac{1}{(2n+1)(2n+3)}$,

$$\therefore \text{ sum} = -\frac{1}{2(2n+1)} + \text{C or} = \tfrac{1}{2} - \frac{1}{2(2n+1)}.$$

(3). This is Geometric, and Sum $= \dfrac{4^n - 1}{3}$.

22. See p. 113.

QUEEN'S COLLEGE, 1828.

1. Ans. Quotient = 21. 4.

2. Ans. .0198, and .54.

3. Ans. The accumulated dividend on the 106*l*. stock amounts to more than that of the 100*l*. stock by 1*l*. 4*s*. $9\frac{1}{2}$*d*. $\frac{2}{5}$.

4. Since $(2m+1)^2 - 1 = 4m^2 + 4m = 4m(m+1)$, and either m, or $m+1$, must be even,

$$\therefore (2m+1)^2 - 1 \text{ is divisible by } 8.$$

5. Ans. 18. $\frac{118}{119}$ years.

6. Ans. 1828 in the septenary scale is 5221.

7. Since $\sqrt[4]{-1} = \sqrt{(0 + \sqrt{-1})}$, the Ans. is $\sqrt{\tfrac{1}{2}}(1 + \sqrt{-1})$.

8. If P be the p^{th} coefficient, then $\text{P}.\ \dfrac{n-p+1}{p}$ is the $(p+1)^{\text{th}}$;

$$\therefore \text{ their difference is } \delta = \frac{\text{P}}{2p}\left(p - \frac{n+1}{2}\right) \therefore \text{ \&c.}$$

See also *Private Tutor*, vol. i. p. 260.

9. $$\left(\frac{1+x}{1-x}\right)^n = \left(\frac{1-x}{1+x}\right)^{-n} = \left(1 - \frac{2x}{1+x}\right)^{-n},$$
$$= 1 + n.\ \frac{2x}{1+x} + n.\ \frac{n+1}{2}.\ \frac{2^2\,x^2}{(1+2)^2} + \text{ \&c.}$$

10. Since $a : b :: 1 : \dfrac{n-p}{p+1}.\ \dfrac{n-p-1}{p+2}$, we get
$$n = p + \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\left\{1 + \frac{4b}{a}(p+1)(p+2)\right\}}.$$
And, in this particular case, $n = 12$.

11. Ans. $\{n-(p-1)\}\ (n-p)+1$.

12. If $a+ar^n$, and ar^m+ar^{n-m} be the respective sums, and δ their difference; then
$$\delta = a\ (1-r^n)\ (1-r^{n-m}),$$
which is always positive, when n is $> m$.

13. Let $2n$ be the number of terms, a the first term, r the ratio, s the sum of the odd terms, and s' that of the even terms; then
$$s = a + ar^2 + ar^4 + \ldots \ n \text{ terms},$$
$$= a.\ \frac{r^n - 1}{r - 1},$$
$$s' = ar + ar^3 + \ldots \ n \text{ terms},$$
$$= ar.\ \frac{r^n - 1}{r - 1},$$
$$\therefore\ s : s' :: 1 : r.$$

14. (1). Ans. Sum $= 0$.
(2). Ans. Sum of 12 terms $= \frac{4097}{1536}$, and to $\infty = \frac{8}{3}$.

(3). Ans. by Method of Increments; the

$$\text{sum} = \frac{(2n+1)(2n+3)(2n+5)}{6} - \tfrac{5}{2}. \quad (\textit{Wood}, \text{ art. } 420.)$$

(4). Ans. $\dfrac{3n^4+10n^3+9n^2+2n}{12}$ (by Increments).

15. Increment of the sum S of the series, is

$$\Delta\, S = A_n.\, A_{n+1}.$$

$$\therefore S = \frac{A_{n-1}.\, A_n.\, A_{n+1}}{3b} + \text{correction},$$

$$= \frac{A_{n-1}.\, A_n.\, A_{n+1} - A_0.\, A_1.\, A_2}{3b},$$

b being the common difference of the progression.

16. (1). Ans. $x = 8$.

(2). Ans. $x = 7$.

(3). Ans. $x = \frac{3}{2}$, and $\frac{3}{10}$.

(4). Ans. $x = 5$, and -2,

(5). Ans. $\left.\begin{array}{l} x = 7 \\ y = 2 \end{array}\right\}$,

(6). Ans. $\left.\begin{array}{l} x = 49 \\ y = 25 \end{array}\right\}$, and $\left.\begin{array}{l} x = 25 \\ y = 49 \end{array}\right\}$.

17. Since $\sqrt{(12-4\sqrt{3})} = 2\sqrt{3}\sqrt{(\sqrt{3}-1)}$

$$= 2\sqrt{3}\left\{\sqrt{\frac{\sqrt{3}+\sqrt{2}}{2}} - \sqrt{\frac{\sqrt{3}-\sqrt{2}}{2}}\right\}$$

$$= \sqrt{6}.\ \{\sqrt{(\sqrt{3}+\sqrt{2})} - \sqrt{(\sqrt{3}-\sqrt{2})}\},$$

the extraction, as required, is not feasible. The question is most probably misprinted in the paper.

The required cube root is $\sqrt{3}+\sqrt{2}$.

18. The roots are 1, -1, 3 and 4, and are found by assuming them of the form a, $\dfrac{12}{a}$, b, c, and making use of the coefficients -7, -12 and 11.

19. Supposing the roots a, $-a$, b, c, we have

$$b+c=p,\ bc-a^2 = q, \text{ and } a^2bc = s.$$

Whence $bc = \dfrac{q+\sqrt{(q^2-4s)}}{2}$,

and $\therefore\ 2b = p+\sqrt{\{p^2-2q-2\sqrt{(q^2-4s)}\}}$,

$$2c = p-\sqrt{\{p^2-2q-2\sqrt{(q^2-4s)}\}},$$

and $a = \sqrt{\dfrac{\sqrt{(q^2-4s)}-q}{2}}$.

20. Ans. $A = -\frac{1}{4}$, $B = -\frac{3}{16}$, $C = \frac{2}{5}$, and $D = -\frac{17}{80}$.

21. Log $512 = \log 2^9 = 9\log 2 = 2.70927$.

22. Ans. $x = \dfrac{\log\{b+\sqrt{(k+b^2)}\}}{\log a}$.

23. Since $5 = \dfrac{7\times 2+1}{3}$,

$$9 = \frac{7\times 2^2-1}{3},$$

$$19 = \frac{7\times 2^3+1}{3}, \text{ \&c.}$$

$\therefore$ the n^{th} term of the series is

$$\frac{7\times 2^{n-1}+(-1)^n}{3},$$

$\therefore$ the series consists of two Geometric Series, whose general terms are

$$\tfrac{7}{3}\times 2^{n-1}, \text{ and } \frac{(-1)^n}{3},$$

whose common ratios are

$$2, \text{ and } (-1),$$

and first terms

$$\tfrac{7}{3}, \text{ and } -\tfrac{1}{3}.$$

Also, the sum of n terms is

$$\frac{7\times 2^{n+1}-15-(-1)^{n-1}}{6}.$$

Observe. The series is recurring; its scale of relation being $1+2$.

24. Ans. $x = 7$, and $7 = 5$.

QUEEN'S COLLEGE, 1830.

1. Ans. 2*l*. 10*s*. 9*d*.

2. Ans. $\frac{17}{5280}$,

3. Ans. 1*s*. 6*d*. a day.

4. Ans. 25*l*.

5. At simple interest

$$\text{present value} = \frac{P}{1+nr} \cdot (\textit{Wood}, \text{art. } 392).$$

At compound interest

$$\text{present value} = \frac{P}{R^n} \cdot (\textit{Wood}, \text{art. } 398).$$

Hence, in the former case,

$$\text{P. value} = \frac{875.\ 9.\ 6}{1+\frac{5\frac{1}{2}\times 3\frac{1}{2}}{100}} = 734l.\ 3s.\ 0d.\ \frac{42}{53},$$

and, in the latter,

$$\text{P. value} = \frac{875.\ 9.\ 6}{\left(1+\frac{3\frac{1}{2}}{100}\right)^{\frac{11}{2}}} = \frac{35019\times 10^{10}}{4\times 207^{\frac{11}{2}}} l.$$

which may easily be completed by reference to Logarithmic Tables.

$$\begin{aligned} \text{For log P. value} &= 10+\log 35019-2-\tfrac{11}{2}.\log 207, \\ &= 8+\log 35019-\tfrac{11}{2}.\log 207. \end{aligned}$$

6. Ans. See *Wood*, for the proof.

The quotient $= 12500$.

7. The quotients are $x-\sqrt{x}$, and $(x-2)(x-4)$.

8. Ans. The product $= \dfrac{a^4-x^4}{a^2x}$.

9. See *Wood*, art. 92. The quantities have no common measure.

10. (1.) Ans. $x = 7$.

(2). $x = 2$, and $-\frac{3}{7}$.

(3). $\left.\begin{array}{l} x = ac + b^2 \\ y = \dfrac{ab - c}{a^2 + b} \end{array}\right\}.$

(4). To the first add $2 \times xy$; thence

$$x + y = 5, \text{ and } -6.$$

Square this, and subtract $4xy$, &c., then

$$x - y = \pm 1, \text{ and } \pm \sqrt{12},$$
$$\therefore\ 2x = 6, \text{ and } 4 \text{ also } -6 \pm \sqrt{12},$$
$$\text{and } 2y = 4, \text{ and } 6 \text{ also } -6 \mp \sqrt{12},$$

$\therefore$ the simultaneous values are

$$\left.\begin{array}{l} x = 3 \\ y = 2 \end{array}\right\} \left.\begin{array}{l} x = 2 \\ y = 3 \end{array}\right\} \left.\begin{array}{l} x = -6 + \sqrt{12} \\ y = -6 - \sqrt{12} \end{array}\right\} \left.\begin{array}{l} x = -6 - \sqrt{12} \\ x = -6 + \sqrt{12} \end{array}\right\},$$

(5). From the second

$$x^2 + 2xy + y^2 = b^2 - 2b\sqrt{(xy)} + xy,$$
$$\therefore\ \sqrt{(xy)} = \frac{b^2 - a}{2b},$$
$$\text{and } xy = \frac{(b^2 - a)^2}{4b^2}$$

Add this to the first of the given equations, which will give $x + y$; and subtract $3xy = 3.\dfrac{(b^2 - a)^2}{4b^2}$ from that equation, and it will give $x - y$. Whence x and y.

(6). Squaring the second, we get, by means of the third,

$$xy + xz + yz = 56,$$
$$\therefore\ xy + y^2 + yz = 56,$$
$$\therefore\ y = \frac{56}{x + y + z} = \frac{56}{14} = 4.$$

Hence $x^2 + z^2 = 84 - 16 = 68$,

$$\text{and } 2xz = 32,$$
$$\therefore\ x + z = \pm 10,$$
$$\text{and } x - z = \pm 6.$$

11. Let x = number he purchased; then they cost him $\dfrac{240}{x}$ a head, and he sold the $x - 3$ oxen at $\dfrac{240}{x} + 8$ a head,

∴ by the question

$$(x - 3)\left(\frac{240}{x} + 8\right) = 240 + 59 = 299,$$

from which, by the solution of a quadratic, it is found, that

$$x = 16.$$

12. The sum of the first series $= - 275$,

that of the second $= \frac{3}{10}$.

13. Let x be the first term, y the common difference; M the m^{th} term, N the n^{th} term; then

$$x + (m - 1)\,y = \text{M},$$
$$x + (n - 1)\,y = \text{N},$$
$$\therefore\ y = \frac{\text{M} - \text{N}}{m - n},$$

and $\therefore\ x = \text{M} - (m - 1).\,\dfrac{\text{M} - \text{N}}{m - n} = \dfrac{(m-1)\,\text{N} - (n-1)\text{M}}{m-n}$.

Hence, the sum of p terms of the series is

$$\left\{2.\,\frac{(m-1)\text{N} - (n-1)\text{M}}{m-n} + \frac{\text{M} - \text{N}}{m-n}.\,(p - 1)\right\}\frac{p}{2},$$

or, $\{(2m - p - 1)\,\text{N} - (2n + p + 1)\,\text{M}\}\,\dfrac{p}{2(m-n)}$.

14. See *Private Tutor*, 199, vol. 1.

The second form is proved by expanding the Binomial according to the Theorem, and then collecting the terms equally distant from either extreme.

15. $\sqrt{\{1 + \sqrt{(1 - m^2)}\}} = \sqrt{\dfrac{1 + m}{2}} + \sqrt{\dfrac{1 - m}{2}}$.

16. The base of any number N is a in the equation

$$\text{N} = a^x,$$

in which x is defined to be the logarithm of N in the system whose base is a.

For the determination of the base of the hyperbolic system, see *Private Tutor*, vol. i. pp. 24, 25.

$$\textit{Nap. log}\ (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

$$\therefore \text{ Nap. } \log (1-x) = x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots$$

$$\therefore \text{ Nap. } \log \frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \&c.\right\},$$

$$\text{Let } \frac{1+x}{1-x} = 1 + \frac{1}{c},$$

$$\text{thence } x = \frac{1}{2c+1},$$

$$\text{and } \log(1+c) = \log c + 2\left\{\frac{1}{2c+1} + \frac{1}{3}\cdot\frac{1}{(2c+1)^3} + \&c.\right\}.$$

17. Let $x^2 + y^2 = z^2$, and make $z = nx + y$; then

$$z^2 = n^2x^2 + 2\,nxy + y^2,$$

$$\therefore y = \frac{1-n^2}{2n}\cdot x,$$

$$\text{and } \therefore x^2 + y^2 = x^2\left\{1 + \frac{(1-n^2)^2}{4n^2}\right\}\&c.$$

$$= x^2\cdot\frac{(1+n^2)^2}{4n^2},$$

which is a square, whatever may be the assumed values of n and x.

18. See p. 46, No. 7.—p. 59, No. 6,—p. 61, No. 19.

19. This is a recurring series, whose scale of relation is

$$f = 13,\ g = -44 \text{ and the}$$

$$\text{Sum to } \infty = ar.\frac{1-8\,ar}{1-13ar+44a^2r^2}.$$

To ascertain the n^{th} term of the series, let $C_1, C_2 \ldots C_n$ be the coefficients of the successive terms; then, making $a = 13$, and $b = 44$, we have

$C_3 = aC_2 - bC_1 = aC_2 - b,$

$C_4 = aC_3 - bC_2 = (a^2 - b)\,C_2 - ab,$

$C_5 = (a^3 - 2ab)\,C_2 - a^2b + b^2,$

$C_6 = (a^4 - 3a^2\,b + b^2)\,C_2 - a^3\,b + 2ab^2,$

$C_7 = (a^5 - 4a^3\,b + 3ab^2)\,C_2 - a^4b + 3a^2\,b^2 - b^3,$

$C_8 = (a^6 - 5a^4\,b + 6a^2\,b^2 - b^3)\,C_2 - a^5b + 4a^3b^2 - 4ab^3,$

$C_9 = (a^7 - 6a^5\,b + 10a^3b^2 - 4ab^3)C_2 - a^6\,b + 5a^4\,b^2 - 6a^2\,b^3 + b^4,$

$$C_{10} = (a^8 - 7a^6 b + 15a^4 b^2 - 10a^2 b^3 + b^4) C_2 - a^7 b + 6\, a^5 b^2 - 10\, a^3 b^3 + 4ab^4, \text{ \&c.}$$

and, generally,

$$C_n = \left\{ a^{n-2} - (n-3)\, a^{n-4}\, b + \frac{(n-5)\,(n-4)}{1.\ 2} a^{n-6}\, b^2 - \frac{(n-7)\,(n-6)\,(n-5)}{1.\quad 2.\quad 3} a^{n-8}\, b^3 + \text{\&c.} \right\} C_2$$
$$- a^{n-3}\, b + (n-4)\, a^{n-5}\, b - \frac{(n-6)\,(n-5)}{1.\ 2} a^{n-7}\, b^3 + \frac{(n-8)\,(n-7)\,(n-6)}{1.\quad 2.\quad 3} a^{n-9}\, b^3 - \text{\&c.}$$

whence may be found any term of the series, a being 13 and $b = 44$.

But the sum of the series to infinity, commencing from the ($n+1^{\text{th}}$) term, is (*Wood*, art. 414),

$$\frac{C_{n+1}\, a^{n+1}\, r^{n+1} + C_{n+2}\, a^{n+2}\, b^{n+2} - 13ar.\ C_{n+1}\, a^{n+1}\, b^{n+1},}{1 - 13ar + 44a^2 r^2},$$

and, consequently, this sum from that of the whole series to ∞, we shall have the sum of n terms of the series.

20. Let $a - b$, a, $a + b$, be the roots; then

$$3a = 3 \text{ and } \therefore a = 1.$$

But this does not verify the equation; consequently the roots are not in Arithmetic progression.

21. See p. 94, and *Wood*, art. 341.

22. *Wood*, art. 352.

CORPUS CHRISTI COLLEGE, 1827.

1. The duty is $\frac{41}{400} \times (2377 + \frac{1}{5} + \frac{1}{80}) \div 5763$ per bushel,

$$\text{or, } 10\frac{109157}{768400} \text{ pence.}$$

2. Amount $\propto$ number of men $\times$ number of days $\times$ wages per day,

$$\therefore \text{number of men} \propto \frac{\text{amount}}{\text{number of days} \times \text{wages per day}},$$

and if x be the number of men required; then

$$x : \frac{103\frac{1}{2}}{24 \times \frac{1}{2}} : : 25 : \frac{76\frac{2}{3}}{16},$$

which gives $x = 45$.

3. $i - i_1$ is the sum paid yearly as premium for the assurance;

$\therefore \dfrac{i - i_1}{a} \times 100$ is the sum his executors will claim at his decease; the annual interest of this is

$$\frac{i - i_1}{a} \times 100 \times \frac{r}{100} = i_1 \text{ by the question,}$$

$$\text{whence } i_1 = \frac{ri}{a + r}.$$

4. *Wood*, art. 20 and 377.

5. The G. C. M. is $x^2 + 7$,

and the L. C. M. is 36.

6. This is evident from the continued multiplication of $(a \pm b)(a \pm b)(a \pm b) \ldots$, to n factors.

Those who are not satisfied with this *sufficient* proof, may consult *Barlow's* Theory of Numbers on the subject.

7. See *Private Tutor*, vol. i. p. 262.

8. Let x, y, z be the gallons per minute discharged by the first, second and third spouts respectively; then

$$\left.\begin{aligned}(x+y)\,32 &= 384\\ (x+z)\,24 &= 384\\ \text{and } (x+y+z)\,16 &= 384\end{aligned}\right\},$$

$$\therefore x = 4, y = 8, z = 12,$$

9. (1). $x = 13, y = 3$.

(2). $\dfrac{x+1}{x} = \frac{3}{2}$ and $\frac{1}{2}$ $\therefore x = 2$ and -3.

10. $x = 8$ and $y = 4$.

11. (1). The four values of x are $\pm \sqrt{(1 \pm \sqrt{\frac{1}{2}})}$.

(2.) The first equation (the mistake is in the original paper) should have been

$$\sqrt{\frac{3x}{x+y}} - \sqrt{\frac{x+y}{3x}} = 2,$$

From this $\frac{x+y}{3x} = 3 \pm 2\sqrt{2}$,

$$y = (8 \pm 6\sqrt{2})\,x,$$

$$\therefore\ (8 \pm 6\sqrt{2})\,x^2 - (9 \pm 6\sqrt{2})\,x = 54,$$

$$\therefore\ x^2 - \frac{9 \pm 6\sqrt{2}}{8 \pm 6\sqrt{2}}\,x = \frac{54}{8 \pm 6\sqrt{2}},$$

$$\therefore\ x = \tfrac{3}{4}.\frac{3 \pm 2\sqrt{2} \pm \sqrt{209 \pm 156\sqrt{2})}}{4 \pm 3\sqrt{2}},$$

which comprises four different values, each of which may easily be computed in decimals, after first rationalising the denominator.

12. (1). The common difference $= -\frac{1}{3}$,

$$\text{and the sum} = \left(1 - \frac{19}{3}\right)10 = \frac{-160}{3}.$$

(2). The common ratio is $\frac{2}{3}$,

$$\text{and sum} = \frac{\frac{1}{2}}{\frac{1}{3}} = \tfrac{3}{2}.$$

(3). The series is of the form

$$x\,(1 + 3x + 5x^2 + \&c.)$$

which is recurring its scale of relation being $f+g = 2-1$,

$\therefore$ the sum of n terms

$$= x.\frac{1 + x - (2n+1)\,x^n + (2n-1)\,x^{n+1}}{(1-x)^2}, \text{ (}\textit{Wood}\text{, art. 414.)}$$

$$= \tfrac{1}{2}.\frac{\frac{3}{2} - \frac{2n+1}{2^n} + \frac{2n-1}{2^{n+1}}}{(\frac{1}{2})^2},$$

$$= 3 - \frac{2n+1}{2^{n-1}} + \frac{2n-1}{2^n},$$

$$= \frac{3.\ 2^n - 2n - 3}{2^n}.$$

13. Let S $= qprrr$ . . . . then

$$10^{l+m+n}\ \mathrm{S} = qpr.\,rrr\ .\ .\ .$$

$$\text{and } 10^{l+m}\ \mathrm{S} = qp.\,rrr\ .\ .\ .\ .$$

$$\therefore S = \frac{qp.\ (r-1)}{10^{l+m}\ (10^n-1)} = \frac{qp\ (r-1)}{999\ldots n \text{ nines } 0000\ldots (l+m) \text{ zeros}}.$$

14. Let $\dfrac{1}{a+b}$, $\dfrac{1}{a+2b}$, $\dfrac{1}{a+3b}$, &c. be the Harmonic series; this form being assumed, because the reciprocals of quantities in Harmonic, are in Arithmetic progression.

Now any two pairs of consecutive terms are of the form

$$\frac{1}{a+nb},\ \frac{1}{a+(n+1)b},$$

$$\text{and } \frac{1}{a+mb},\ \frac{1}{a+(m+1)b},$$

$$\text{and } \frac{1}{a+nb} - \frac{1}{a+(n+1)b} = \frac{b}{(a+nb)\ \{a+(n+1)b\}},$$

$$\frac{1}{a+mb} - \frac{1}{a+(m+1)b} = \frac{b}{(a+mb)\ \{a+(m+1)b\}},$$

whence the proposition is manifest.

15. Let x and y be the two numbers; then

$$\left.\begin{aligned} \frac{x+y}{2} &= \sqrt{(xy)}+13 \\ \text{and } \sqrt{(xy)} &= \frac{2\,xy}{x+y} + 12 \end{aligned}\right\}.$$

Hence, immediately

$$\left.\begin{aligned} x+y &= 338 \\ \text{and } 2\sqrt{xy} &= 312 \end{aligned}\right\} \therefore \left.\begin{aligned} \sqrt{x}+\sqrt{y} &= \pm\sqrt{650} \\ \sqrt{x}-\sqrt{y} &= \pm\sqrt{\ 26} \end{aligned}\right\},$$

whence x and y.

16. $\sqrt{(2\sqrt{-1})} = \sqrt{(0+2\sqrt{-1})} =$ by the rule $1+\sqrt{-1}$.

$$\sqrt[3]{(11+5\sqrt{7})} = \frac{1+\sqrt{7}}{\sqrt[3]{2}}.$$

17. See *Private Tutor*, vol. i. p. 262.

18. Transform the number 1319 to the ternary scale; in that scale it is 1210212,

$$\therefore\ 2\times 1+1\times 3+2\times 3^2+1\times 3^4+2\times 3^5+1\times 3^6 = 1319.$$

19. Adding the two equations together, we get

$$3x - y = 12,$$

$$\therefore\ y = 12 - 3x,$$

$$\therefore \text{ making } x = 0,\ \pm 1,\ \pm 2,\ \pm 3, \text{ \&c.}$$

$$\text{the values of } y = 12,\ 12 \mp 3,\ 12 \mp 6,\ 12 \mp 9, \text{ \&c.}$$

$$= 12,$$

$$\text{and } x = \ 9,\ 6,\ 3,\ 0,\ -3,\ -6, \text{ \&c.}$$

$$y = 15,\ 18,\ 21,\ 24,\ 27,\ 30, \text{ \&c.}$$

$$\text{Hence } z = 5 + 2y - x = 29 - 6x,$$

$$= 29,\ 29 \mp 6,\ 29 \mp 12,\ 29 \mp 18, \text{ \&c.}$$

$$= 29,\ 23,\ 17,\ 11,\ 5,\ -1,\ -7, \text{ \&c.}$$

$$35,\ 41,\ 47,\ 53,\ 59,\ 65, \text{ \&c.}$$

Whence the simultaneous values of x, y, z are

$$\left.\begin{array}{l} x = \ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7 \ldots \\ y = 12,\ 9,\ 6,\ 3,\ 0,\ -3,\ -6,\ -9 \ldots \\ z = 29,\ 23,\ 17,\ 11,\ 5,\ -1,\ -7,\ 13 \ldots \end{array}\right\},$$

$$\text{and } \left.\begin{array}{l} x = -1,\ -2,\ -3,\ -4,\ -5,\ -6 \ldots\ldots \\ y = \ 15,\ 18,\ 21,\ 24,\ 27,\ 30 \ldots\ldots \\ z = \ 35,\ 41,\ 47,\ 53,\ 59,\ 65 \ldots\ldots \end{array}\right\}.$$

20. See *Private Tutor*, vol. i. p. 24.

CORPUS CHRISTI COLLEGE, 1828.

1. $\frac{2}{5}$ of 2. 6 = 1 shilling. $\therefore$ the Ans. is $\dfrac{1}{13\frac{1}{3}}$ or $\dfrac{3}{40}$.

2. The cost is $650 \times \dfrac{90\frac{1}{2}}{100} = \dfrac{13}{4} \times 181 = 588l.\ 5s.$

3. No. of days $\propto \dfrac{\text{length} \times \text{breadth} \times \text{depth}}{\text{men} \times \text{hours}}$;

$\therefore$ if x be the days required, we have,

$$x : \frac{420 \times 5 \times 3}{24 \times 9} :: 5 : \frac{230 \times 3 \times 2}{248 \times 11},$$

$$\therefore x = \frac{25 \times 42}{24 \times 9} \times \frac{248 \times 11}{23 \times 2},$$

$$= 288 \text{ days } 2\frac{13}{23} \text{ hours.}$$

4. Let P be the present value; then in n years the amount of P is $M=P+nrP$, (*Wood,* art. 391.) and the amount of the annuity in that time is

$$a+2a+3a+ \ldots\ldots n \text{ terms}$$
$$+r\,\{a+3a+6a+10a+\ldots\ldots(n-1) \text{ terms}\}$$
$$=\{2a+a.\,(n-1)\}\,\frac{n}{2}+ar.\,\frac{(n-1)\,n.\,(n+1)}{1.\ 2.\ 3},$$
$$\therefore P=\frac{an\,(n+1)}{6\,(1+nr)}.\ \{3+r\,(n-1)\}.$$

5. $5x^2-1$ is the G. C. M. required.

6. See *Wood,* for the proofs.

The quotient is 12450.

7. (1). Multiply by $\sqrt{(x+\sqrt{x})}$; then

$$x+\sqrt{x}-\sqrt{(x^2-x)}=\tfrac{3}{2}\sqrt{x}.$$

Whence $x=\frac{25}{16}$.

(2). Multiply the first of these by 60, and the second by 24, reduce, &c., and there results

$$\left.\begin{aligned}32x-15y&=84\\12x-13y&=-46\end{aligned}\right\},$$

whence x and y are easily found.

8. (1). $\sqrt{(a+x)}+\sqrt{(a-x)}=\frac{x}{\sqrt{6}}$,

$$\therefore 2a+2\sqrt{(a^2-x^2)}=\frac{x^2}{6},$$
$$\therefore a^2-x^2=\left(\frac{x^2}{12}-a\right)^2=\frac{x^4}{144}-6ax^2+a^2,$$
$$\therefore \frac{x^4}{144}-(6a-1)\,x^2=0,$$

$\therefore x=0$; but this does not satisfy the equation.

Again, $x^2=144.\,(6a-1)$,

$$\therefore x=\pm 12\sqrt{(6a-1)}.$$

(2). From the first,

$$\left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)=\frac{19}{4},$$

$$\therefore \frac{x}{y}+\frac{y}{x} = \pm \sqrt{5},$$

$$\therefore x^2+y^2 = \pm \sqrt{5}.\, xy.$$

$$\text{But } x^2+y^2 = 4+2xy,$$

$$\therefore 2xy = 8\,(2 \pm \sqrt{5}),$$

$$\text{and } x^2+y^2 = 4+8\,(2\pm\sqrt{5}).$$

$$\therefore (x+y)^2 = 4+16\,(2\pm\sqrt{5}),$$

$$= 36\pm16\sqrt{5} = 4\,(9\pm4\sqrt{5}),$$

$$\therefore x+y = \pm\, 2\,(\sqrt{5}\pm2),$$

$$= 4 \pm 2\sqrt{5}.$$

$$\text{But } x-y = 2,$$

$$\therefore x = 3\pm2\sqrt{5},\ y = 1\pm2\sqrt{5}.$$

9. Let x, y, z be the numbers; then, by the question,

$$\left.\begin{array}{l} x\,(y+z) = 26 \\ y\,(x+z) = 50 \\ z\,(x+y) = 56 \end{array}\right\}.$$

Adding them together, we get

$$xy+xz+yz = 66,$$

$$\therefore \left.\begin{array}{l} yz = 40 \\ xz = 16 \\ xy = 10 \end{array}\right\}.$$

Multiply these together, and there results,

$$x^2y^2z^2 = 400\times16,$$

$$\therefore xyz = \pm\, 80,$$

$$\therefore x = \pm\, 2,\ y = \pm\, 5,\ z = \pm\, 8.$$

10. $S_1=\{2+(n-1)\}\frac{n}{2}$, $S_2=\{4+(n-1).\,3\}\,\frac{n}{2}$,

$$S_3 = \{6+(n-1)\,5\}\,\frac{n}{2} + \&c.$$

$$\therefore S_1+S_2+S_3+\ldots S_p=\frac{n}{2}\times p + \frac{n^2}{2}(1+3+5+\ldots p \text{ terms}),$$

$$=\frac{np}{2}+\frac{n^2}{2}\,\{2+2\times(p-1)\}\,\frac{p}{2},$$

$$=\frac{np}{2}+\frac{n^2}{2}\,p^2=\frac{np}{2}\,(np+1).$$

11. See p. 66.

12. Let $S = \frac{a}{r} + \frac{a+b}{r^2} + \frac{a+2b}{r^3} + \ldots n$ terms,

$$= \frac{a}{r} + \frac{a}{r^2} + \frac{a+b}{r^3} + \ldots n \text{ terms,}$$

$$+ \frac{b}{r^2} + \frac{b}{r^3} + \ldots (n-1) \text{ terms,}$$

$$= \frac{a}{r} \cdot \frac{b}{r^2} \cdot \frac{\left(\frac{1}{r}\right)^{n-1} - 1}{\frac{1}{r} - 1} + \frac{1}{r} \cdot \left\{\frac{a}{r} + \frac{a+b}{r^2} + \ldots (n-1) \text{ terms}\right\}$$

$$= \frac{a}{r} + \frac{b}{r^n} \cdot \cdot \frac{1 - r^{n-1}}{1-r} + \frac{1}{r} \cdot \left\{S - \frac{a+(n-1)b}{r^n}\right\},$$

$$\therefore S.\,(r-1) = a + \frac{b}{r^{n-1}} \cdot \frac{r^{n-1} - 1}{r-1} - \frac{a+(n-1)b}{r^n},$$

$$= a\,\frac{r^n - 1}{r^n} + \frac{b}{r^{n-1}} \cdot \left(\frac{r^{n-1} - 1}{r-1} - \frac{n-1}{r}\right),$$

$$= a\,\frac{r^n - 1}{r^n} + \frac{b}{r^n . (r-1)} \cdot (r^n - nr + n - 1),$$

$$\therefore S = a.\,\frac{r^n - 1}{r^n\,(r-1)} + b.\,\frac{r^n - nr + n - 1}{r^n\,(r-1)^2}.$$

To apply this form to the numerical example, we have

$$a = 1,\ b = 2, \text{ and } r = 3,$$

$$\therefore S = \frac{3^n - 1}{2 \times 3^n} + \frac{3^n - 2\,n - 1}{2 \times 3^n},$$

$$= \frac{3^n - n - 1}{3^n},$$

13. See *Private Tutor*, vol. i. p. 262.

14. The number is $\dfrac{1.\ 2.\ 3.\ 4 \ldots .\ m}{1.\ 2.\ 3 \ldots p}$.

Thus, the factors of $a^3.\ b.\ c.$ may have

$$\frac{1.\ 2.\ 3.\ 4.\ 5.}{1.\ 2.\ 3} = 20$$

different arrangements.

15. Let x be the common ratio; then

$$x^n = x^{n+1} \times \frac{1}{1-x},$$

$$\therefore \frac{x}{1-x} = 1,$$

$$\text{and } x = \tfrac{1}{2},$$

$$\therefore \text{ the series is } 1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots \infty.$$

16. See *Wood*, art. 259.

$$\sqrt[3]{(2\sqrt{7}+3\sqrt{3})} = \tfrac{1}{2}(\sqrt{7}+\sqrt{3}).$$

17. Let ax be the number, and make $\frac{ax}{b} = y + \frac{c}{b}$,

$$\text{then } ax - by = c,$$

in which integer values of x and y will give the Answer.

In the particular instance,

$$10x - 17y = 9,$$

$$\therefore x = y + \frac{7y+9}{10}.$$

$$\text{Make } 7y + 9 = 10w,$$

$$\therefore y = w - 1 + \frac{3w-2}{7}.$$

$$\text{Again, make } 3w - 2 = 7v,$$

$$\therefore w = 2v + \frac{v+2}{3}.$$

$$\text{Make } v + 2 = 3u,$$

$$\therefore v = 3u - 2,$$

$$\text{in which } u \text{ may} = 1, 2, \&c.$$

$$\text{Hence, } 3w = 21u - 12,$$

$$\therefore w = 7u - 4,$$

$$\therefore y = 10u - 7,$$

$$\text{and } x = 10u - 7 + \frac{70u - 40}{10},$$

$$= 17u - 11,$$

$$\therefore ax = 170u - 110,$$

which is exactly divisible by 10, and whatever is the value of u when ax is divided by 17, it will always leave the remainder 9; and these numbers are

60, 230, 400, &c.

18. If possible, let

$$\frac{(m\,\phi)^2}{a} \text{ and } \frac{(n\,\phi)^2}{a},$$

have the same remainder r; then if q, q' are the quotients,

$$(m\,\phi)^2 = qa + \text{R},$$
$$(n\,\phi)^2 = q'a + \text{R},$$
$$\therefore q - q' = \frac{\phi^2\,(m^2 - n^2)}{a} \text{ an integer.}$$

But ϕ is prime to a, and $\therefore\ \phi^2$ is.

Hence, $m^2 - n^2$ is divisible by a; that is, $(m-n)(m+n)$ is divisible by a. But a is prime, and $\therefore$ either $m-n$ or $m+n$ is divisible by a. But each of them is less than a

$$\therefore \frac{(m\phi)^2}{a} \text{ and } \frac{(n\phi)^2}{a}$$

cannot have the same remainder.

19. See *Wood*, art. 383, or *Woodhouse's* Trigonometry, p. 211.

CORPUS CHRISTI COLLEGE, 1830.

1. His tax amounts to

$$\left(1384 + \tfrac{3}{4}\right) + \left(\frac{1}{10} + \frac{19}{480}\right),$$

$$\text{or } \frac{5539 \times 67}{1920} \text{ or } 193l.\ 5s.\ 9\tfrac{1}{8}d.$$

$\therefore$ the net income is $1191l.\ 10s.\ 2\tfrac{7}{8}d.$

2. Their Least Common Denominator is 90, and the sum of the fractions is

$$\frac{36+60+50+63}{90} \text{ or } \frac{209}{90}.$$

3. The quotient

$$= \frac{510000}{342537456} = \frac{42500}{28544788},$$
$$= \frac{10625}{7136197} = .0014, \&c.$$

$\sqrt{25405534}$ in the senary $= 1503.51$ nearly.

It does not terminate,

$\frac{7}{9}$ cwt. $= 3\frac{1}{9}$ quarters $=$ 3q. 3lb. $\frac{1}{9}$ $=$ 3q. 3lb. $1\frac{7}{9}$ oz.
.42857 month = 1 week, 4 days, 23 hours, 59 minutes, 56 seconds.

4. The area of the field $= 52\dfrac{2}{3} \times 34\dfrac{3}{4}$ or $\dfrac{10981}{6}$ square ft.

$\therefore$ the price $= \dfrac{10981}{6} \times \dfrac{25}{20}l. = \dfrac{54905}{24} = 2287l.\ 14s.\ 2d.$

5. Present Worth $= \dfrac{\text{sum}}{1+nr}$ (*Wood*, art. 392.)

$$= \frac{543+\frac{7}{20}}{1+\frac{130}{365}\times\frac{37}{800}} = \frac{10867}{1+\frac{481}{29200}}$$

$$= \frac{10867\times 1460}{29681} = 534\frac{16166}{29681}\,l.$$

which can be reduced to shillings, pence, &c.

6. See *Wright's* Self-Instructions in Pure Arithmetic on the subject of the Cube Root.

7. See the last quoted work for these subjects in Arithmetic, and *Wood* for them in Algebra.

8. (1). The G. C. M. in the first, being x^2+xy+y^2, the fraction becomes

$$\frac{x^2-xy+y^2}{x^2+xy+y^2},$$

(2). This $= \dfrac{(e^{2x}-1)(x^3+1)}{(e^{2x}+2e^x+1)(x^2-1)},$

$$= \frac{(e^x-1)(e^x+1)(x^2-x+1)(x+1)}{(e^x+1)^2(x-1)(x+1)},$$

$$= \frac{e^x-1}{e^x+1}\cdot\frac{x^2-x+1}{x-1}.$$

9. *Wood*, art. 377.

10. $\dfrac{1}{(a-b)(a-c)(x+a)} - \dfrac{1}{(a-b)(b-c)(x+b)},$

$$= -\frac{1}{a-b} \cdot \frac{(a-b)\,x+a^2-b^2-c\,(a-b)}{(x+a)\,(x+b)},$$

$$= -\frac{x+a+b-c}{(a-c)\,(b-c)\,(x+a)\,(x+b)},$$

$$\therefore \text{ the aggregate } = \frac{1}{(a-c)(b-c)} \times \left\{\frac{1}{x+c} - \frac{x+a+b-c}{(x+a)(x+b)}\right\}$$

$$= \frac{1}{(a-c)\,(b-c)} \times \frac{ab-ac-bc+c^2}{(x+c)\,(x+b)},$$

$$= \frac{1}{(x+a)\,(x+b)}.$$

11. (1). Ans. $x = 13$,

(2). Ans. $x = 20$,

(3). Cubing both sides,

$$2+3a\sqrt[3]{(1-x)} = a^3,$$

$$\therefore 1-x = \left(\frac{a^3-2}{3a}\right)^3$$

$$\therefore x = 1 - \frac{(a^3-2)^3}{27a^3}.$$

(4). Dividing by $(a^n - x^n)^{\frac{2}{m}}$ we get,

$$\left(\frac{a^n+x^n}{a^n-x^n}\right)^{\frac{2}{m}} - \left(\frac{a^n+x^n}{a^n-x^n}\right)^{\frac{1}{m}} = -1,$$

$$\therefore \left(\frac{a^n+x^n}{a^n-x^n}\right)^{\frac{1}{m}} = \frac{1}{2} \pm \frac{\sqrt{3}}{2},$$

$$\therefore a^n + x^n = (a^n - x^n)\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)^m,$$

$$\therefore x^n = a^n \frac{\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)^m - a^n}{1 + \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)^m},$$

$$= a^n \cdot \frac{(1 \pm \sqrt{3})^m - 2^m}{(1 \pm \sqrt{3})^m + 2^m},$$

$$\therefore x = a\left\{\frac{(1 \pm \sqrt{3})^m - 2^m}{(1 \pm \sqrt{3})^m + 2^m}\right\}^{\frac{1}{n}}.$$

(5). Taking the logarithms of both sides,

$$m \log a \times x + n \log b \times y = \log k,$$

$$m' \log a' \times x + n' \log b' \times y = \log k',$$

whence, by the usual method,

$$x = \frac{n' \log k - u \log k'}{mn' \log a . \log b' - m'n \log a' \log b'}$$

$$y = \frac{m' \log k - m \log k'}{m'n \log a' \log b - m\,n' \log a \log b'}.$$

(6). Let the roots be $\frac{a}{r}$ a, ar; then $a^3 = 64 \therefore a = 4$,

and the other roots will be determined by

$\frac{4}{r} + 4 + 4r = 14$, which gives $r = 2$ and $\frac{1}{2}$,

whence the roots are 2, 4, 8.

(7). Let a be the first root; then the others are $2a$, $5a$, and $8a = 16$; $a = 2$, and the other roots are 4 and 10.

12. (*Wood*, art. 224).

$$.32415415 \ldots . = \frac{32}{100} + \frac{415}{99900} = \frac{32383}{99900}.$$

13. *Wood*, art. 228, for the Permutations.

For the Problem, see *Private Tutor*, vol. i. p. 260. ex. 41.

14. $(a - b\,x - cx^2)^{\frac{m}{n}} = a^{\frac{m}{n}}\left\{1 - \left(\frac{b}{a} + \frac{c}{a}\,x\right)x\right\}^{\frac{m}{n}}.$

Of this, the 6th term is,

$$- a^{\frac{m}{n}} . \frac{m}{n} . \frac{\frac{m}{n} - 1}{2} . \frac{\frac{m}{n} - 2}{3} . \frac{\frac{m}{n} - 3}{4} . \frac{\frac{m}{n} - 4}{5} . \left(\frac{b}{a} + \frac{c}{a}\right)^5 x^5,$$

the 5th is,

$$a^{\frac{m}{n}} . \frac{m}{n} . \frac{\frac{m}{n} - 1}{2} . \frac{\frac{m}{n} - 2}{3} . \frac{\frac{m}{n} - 3}{4} . \left(\frac{b}{a} + \frac{c}{a}\,x\right)^4 x^4,$$

the 4th is,

$$- a^{\frac{m}{n}} . \frac{m}{n} . \frac{\frac{m}{n} - 1}{2} . \frac{\frac{m}{n} - 2}{3} . \left(\frac{b}{a} + \frac{c}{a}\,x\right)^3 x^3,$$

which are all the terms in which x^5 will appear. Hence the coefficient required is,

$$-a^{\frac{m}{n}}\left\{\frac{b^5}{a^5}\cdot\frac{m}{n}\cdot\frac{\frac{m}{n}-1}{2}\cdot\frac{\frac{m}{n}-2}{3}\cdot\frac{\frac{m}{n}-3}{4}\cdot\frac{\frac{m}{n}-4}{5},\right.$$

$$-4\,\frac{b^3c}{a^4}\cdot\frac{m}{n}\cdot\frac{\frac{m}{n}-1}{2}\cdot\frac{\frac{m}{n}-2}{3}\cdot\frac{\frac{m}{n}-3}{4},$$

$$\left.+3\,\frac{bc^2}{a^3}\cdot\frac{m}{n}\cdot\frac{\frac{m}{n}-1}{2}\cdot\frac{\frac{m}{n}-2}{3}\right\},$$

$$\text{or } -\frac{m}{n}\cdot\frac{\frac{m}{n}-1}{2}\cdot\frac{\frac{m}{n}-2}{3}\cdot a^{\frac{m-5}{n}}b$$

$$\left\{\frac{\frac{m}{n}-3}{4}\cdot\frac{\frac{m}{n}-4}{5}b^4\left(\frac{m}{n}-3\right)ab^2c+3\,a^2c^2\right\}.$$

15. $\sqrt{(9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15})}=1+\sqrt{3}+\sqrt{5}$.
To effect this, we try

$$\sqrt{\frac{2\sqrt{3}\times2\sqrt{5}}{2\times2\sqrt{15}}}=1,\ \sqrt{\frac{2\sqrt{3}\times2\sqrt{15}}{2\times2\sqrt{5}}}=\sqrt{3},$$

$$\text{and } \sqrt{\frac{2\sqrt{5}\times2\sqrt{15}}{2\times2\sqrt{3}}}=\sqrt{5}.$$

Similarly, if $\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})}$ can be thus extracted, the root is

$$\sqrt{\frac{\sqrt{bc}}{2\sqrt{d}}}+\sqrt{\frac{\sqrt{bd}}{2\sqrt{c}}}+\sqrt{\frac{\sqrt{cd}}{2\sqrt{a}}}.$$

For the other question, see p. 202. No. 17.

16. *Wood*, art. 190.

17. Let P be the number, whose number of digits is p,
Q . . . . q,

then P is $< 10^p$, and Q is $< 10^q$,

$\therefore$ P $\times$ Q is $< 10^{p+q}$;

that is, it cannot consist of $p+q+1$ digits.

Also, P is not $< 10^{p-1}$, and Q is not $< 10^{q-1}$,

$\therefore$ P $\times$ Q is not $< 10^{p+q-2}$, or it cannot have less than $p+q-1$ digits.

18. $$\frac{N^5-N}{12}=\frac{N(N^4-1)}{12}=\frac{N(N-1)(N+1)(N^2+1)}{12}.$$

But $\dfrac{(N+1)\,N\,(N-1)}{1.\ 2.\ 3}$ is an integer,

and $\because$ N is odd, N^2 is odd, and $\therefore$ N^2+1 is even, and divisible by 2,

$\therefore$ N^5-N is divisible by 12.

19. (1). Ans. $6\frac{5}{4}$.

(2). Ans. $\frac{9}{4}$.

(3).

$$\text{Let } S=\frac{a}{x^m}+\frac{a+b}{x^{m+p}}+\frac{a+2b}{x^{m+2p}}+\dots\frac{a+(n-1)b}{x^{m+(n-1)p}},$$

$$\therefore S.\,x^m=a+\frac{a+b}{x^p}+\frac{a+2b}{x^{2p}}+\dots\frac{a+(m-1)b}{x^{(n-1)p}},$$

$$\therefore \frac{S\,x^m}{x^p}=\frac{a}{x^p}+\frac{a+b}{x^{2p}}+\dots\frac{a+(m-2)b}{x^{(n-1)p}}+\frac{a+(m-1)b}{x^{np}},$$

$$\therefore \frac{S\,x^m(x^p-1)}{x^p}=a+\frac{b}{x^p}\left\{1+\frac{1}{x^p}+\frac{1}{x^{2p}}+\dots\frac{1}{x^{(n-2)p}}\right\}$$
$$-\frac{a+(n-1)b}{x^{np}},$$

$$=a+\frac{b}{x^p}\times\frac{\left(\frac{1}{x}\right)^{(n-1)p}-1}{\left(\frac{1}{x}\right)^p-1}-\frac{a+(n-1)b}{x^{np}},$$

$$=a+\frac{b}{x^{(n-1)p}}\,\frac{x^{(n-1)p}-1}{x^p-1}-\frac{a+(n-1)b}{x^{np}},$$

$$=\frac{a(x^{np}-1)}{x^{np}}+\frac{b}{x^{np}}\cdot\frac{x^{np}-n\,x^p+n-1}{x^p-1},$$

$$\therefore S=\frac{1}{x^{m+(n-1)p}}\left\{a.\,\frac{x^{np}-1}{x^p-1}+b.\,\frac{x^{np}-n\,x^p+n-1}{(x^p-1)^2}\right\}.$$

(4). This series is

$$\frac{1}{1.\ 5}+\frac{1}{3.\ 7}+\frac{1}{5.\ 9}+\dots\dots$$

$$\text{Let } S = \tfrac{1}{1} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \dots \infty,$$

$$\therefore S - \tfrac{4}{3} = \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} + \dots \infty,$$

$$\therefore \tfrac{4}{3} = 4\left(\frac{1}{1.\ 5} + \frac{1}{3.\ 7} + \dots \infty\right),$$

$$\therefore \text{series} = \tfrac{1}{3}.$$

20. See p. 42. No. 21. Also, p. 61.

21. If v be any integer whatever, positive, negative, or zero, then $\left.\begin{aligned} x &= 228 - 11v \\ y &= 7v - 4 \end{aligned}\right\}$ will give all the simultaneous values.

Also, $x + y = 4\,(56 - v)$.

$\therefore$ when $v = 56$, $x + y = 0$ is the least.

22. See *Woodhouse's* Trigonometry. Art. on Log.

ALGEBRA.

PART II.

CORPUS CHRISTI COLLEGE, 1830.

[P. 103.]

1. See *Private Tutor*, Alg. pt. ii. p. 15.

2. *Wood*, art. 311.

3. *Private Tutor*, Alg. part ii. p. 111., &c.

4. *Wood*, art. 307.

5. *Wood*, art. 318.

6. (1). *Wood*, art. 325.

(2). Since

$$\begin{aligned} x^6-1 &= (x^3+1)(x^3-1) \\ &= (x^2-x+1)(x+1)(x^2+x+1)(x-1), \end{aligned}$$

$$\therefore \left.\begin{array}{l} x+1=0,\ x-1=0 \\ x^2-x+1=0, \text{ and } x^2+x+1=0 \end{array}\right\},$$

whence the roots are

$$-1,\ 1,\ \frac{1\pm\sqrt{-3}}{2},\ \frac{-1\pm\sqrt{-3}}{2}.$$

(3). The roots are $-2,\ 3\pm\sqrt{3}$. See also No. 6. p. 97., and No. 12. p. 101.

7. See *Private Tutor*, Alg. pt. ii. p. 119.

8. All the roots are imaginary. See p. 81.

9. The expression is (*Wood*, art. 327.)

$$x = \sqrt[3]{\left\{-\frac{r}{2} \pm \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}} + \sqrt[3]{\left\{-\frac{r}{2} \mp \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}},$$

which is essentially of the form,

$$x = \sqrt[3]{(a+b)} + \sqrt[3]{(a-b)},$$

$$\text{or } = \sqrt[3]{(a+b)} - \sqrt{(b-a)} \text{ when } b \text{ is greater than } a.$$

In the first case,

$$\frac{x}{\sqrt[3]{a}} = \sqrt[3]{\left(1 + \frac{b}{a}\right)} + \sqrt[3]{\left(1 - \frac{b}{a}\right)}$$

$$= \sqrt[3]{\left(1 + \frac{b}{a}\right)} \left\{1 + \sqrt[3]{\frac{1 - \frac{b}{a}}{1 + \frac{b}{a}}}\right\}.$$

Make $\frac{b}{a} = \tan^2 \theta$; whence θ is found by logarithms, and

$$\frac{x}{\sqrt[3]{a}} = \sec^{\frac{2}{3}} \theta. \left\{1 + \sqrt[3]{\frac{1 - \tan^2\theta}{1 + \tan^2\theta}}\right\} = \sec^{\frac{2}{3}} \theta. \{1 + (\cos 2\theta)^{\frac{1}{3}}\}.$$

Again, make $(\cos 2\theta)^{\frac{1}{3}} = \tan^2\phi$, which gives ϕ; and

$$\frac{x}{\sqrt[3]{a}} = \sec^{\frac{2}{3}} \theta. \sec^2\phi,$$

$$\text{and } \log x = \tfrac{2}{3} \log \sec \theta + 2 \log \sec \phi + \tfrac{1}{3} \log a.$$

Similarly may the other case be treated, using $\sin^2\theta$, instead of $\tan^2\theta$, as is obvious.

10. Assume

$$\frac{1+2x}{1-x+3x^2} = 1 + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$

then

$$\begin{aligned} 1+2x = 1 + Bx + {} & Cx^2 + Dx^3 + Ex^4 + Fx^5 + \ldots \\ & - x - Bx^2 - Cx^3 + Dx^4 - Ex^5 - \ldots \\ & + 3x^2 + 3Bx^3 + 3Cx^4 + 3Dx^5 + \ldots \end{aligned}$$

$$\therefore B = 3,\ C - B + 3 = 0;\ \therefore C = 0,$$

$$\therefore D + 3B = 0,\ E - D = 0,\ F - E + 3D = 0, \&c.$$

$$\therefore D = -9,\ E = -9,\ F = -9 + 27 = 18, \&c.$$

$$\therefore \frac{1+2x}{1-x+3x^2} = 1+3x-9x^3-9x^4+18x^5+\&c.$$

The scale of relation is $f+g = 1-2$.

11. See p. 102.

12. (1). See *Wood*, p. 352.

$$\begin{aligned}(2).\ S_p \times S_q &= (a^p+b^p+c^p+\ldots)(a^q+b^q+c^q+\ldots)\\ &= a^{p+q}+a^pb^q+a^pc^q+\ldots\\ &\quad+b^{p+q}+a^qb^p+b^pc^q+\ldots\\ &\quad+c^{p+q}+a^qc^p+b^qc^p+\ldots\\ \therefore S_p \times S_q &= S_{p+q}+\Sigma a^pb^q.\end{aligned}$$

$$\therefore \Sigma a^pb^q = \left\{\begin{array}{l} a^pb^q+a^pc^q+\ldots \\ b^pa^q+b^pc^q+\ldots \\ c^pa^q+c^pb^q+\ldots \\ \quad \&c. \end{array}\right\} = S_p S_q - S_{p+q}.$$

Again, multiplying by $S_r = a^r+b^r+c^r+\ldots$ we get

$$\left.\begin{array}{l} a^{p+r}b^q + a^qb^{p+r} + a^{p+r}c^q + \ldots \\ a^{q+r}b^p + a^pb^{q+r} + a^{q+r}c^p + \ldots \\ a^pb^qc^r + a^qb^pc^r + a^qb^rc^r + \ldots \end{array}\right\} = S_pS_qS_r - S_rS_{p+q}.$$

The first line $= S_{p+r}S_q - S_{p+q+r}$ (as is easily seen),

the second $= S_{q+r}S_p - S_{p+q+r}$,

and the third is $\Sigma(a^pb^qc^r)$,

$$\therefore \Sigma(a^pb^qc^r) =$$
$$S_pS_qS_r - S_pS_{q+r} - S_qS_{p+r} - S_rS_{p+q} + 2S_{p+q+r}.$$

13. Let $x = a+u$, u being a small quantity; then

$$\begin{aligned} x^4 &= a^4+4a^3u+\ldots\\ px^3 &= pa^3+3pa^2u+\ldots\\ qx^2 &= qa^2+2qau+\ldots\\ rx &= ra+ru,\\ \therefore \delta &= \delta_1+(4a^3+3pa^2+2qa+r)u \text{ nearly},\\ \therefore u &= \frac{\delta-\delta_1}{4a^3+3pa^2+2qa+r} \text{ nearly.}\end{aligned}$$

But $a^4+pa^3+qa^2+ra = \delta_1$,

$$\therefore a^3+pa^2+qa+r = \frac{\delta_1}{a},$$

$$\therefore u = \frac{\delta - \delta_1}{a(3a^2 + 2pa + q) + \frac{\delta_1}{a}},$$

$\therefore$ &c.

14. See *Wood*, art. 363.

EMMANUEL COLLEGE.

1. The quotient is $x^{m(n-1)} + x^{m(n-2)} + \ldots 1$.

2. The quantities have no Common Measure.

The L. C. M. =
$$(a-x)(a^2+ax+x^2)(a+x) = a^4+a^3x-ax^3-x^4.$$

3. The root is $x - a - \dfrac{a^2}{2x} -$ &c.

4. (1). Ans. $x = 4$.

(2.) Raising the first to the 4th power, &c., it will easily be found, that

$$x^2y^2 + \tfrac{2}{7}a^2xy = \frac{a^4 - b^4}{14},$$

$$\text{whence } xy = -\frac{a^2}{7} \pm \frac{\sqrt{(18a^4 - 14b^4)}}{14}.$$

Squaring the second of the given equations, and subtracting $4 \times xy$, from the result, $x - y$ will be obtained, and thence x and y.

(3). First $\dfrac{3}{5-x} + \dfrac{1}{4-x} = \dfrac{8}{x+2}$.

$$\text{Ans. } x = \frac{27 \pm \sqrt{57}}{8}.$$

(4). $x^2y^2 + 4xy = 96$,

$$\therefore xy = -2 \pm 10 = 8, \text{ and } -12.$$

$$\left.\begin{array}{l}\text{Hence } x-y = \pm 2, \text{ and } \pm 2\sqrt{10} \\ \text{and } x+y = 6\end{array}\right\}.$$

$$\therefore \left.\begin{array}{l} x = 4, 2, 3 \pm 2\sqrt{10} \\ y = 2, 4, 3 \mp 2\sqrt{10}\end{array}\right\}.$$

5. See *Wood*, art. 127;

or, in the equation $x^2+px = q$, assume

$$x^2+px+Q = \left(x + \frac{p}{2}\right)^2 = x^2+px+\frac{p^2}{4}, \therefore Q = \frac{p^2}{4}.$$

6. *Wood*, art. 162.

7. *Wood*, art. 182.

8. *Wood*, art. 190.

9. The sum of $1+3+5+\ldots 2n+1 = 2n.\frac{n}{2} = n^2$.

10. Let r be the common ratio of a, b, c, d, &c.; then

$$b = ar, c = ar^2, d = ar^3, \text{\&c.}$$

$$\therefore \frac{1}{a^2-b^2} = \frac{1}{a^2(1-r^2)}, \frac{1}{b^2-c^2} = \frac{1}{a^2r^2 - a^2r^4} = \frac{1}{a^2r^2(1-r^2)},$$

Hence the series $\frac{1}{a^2-b^2}, \frac{1}{b^2-c^2}$, &c. is Geometric, its common ratio being $\frac{1}{r^2}$, or $\frac{a^2}{b^2}$.

Hence the required sum

$$= \frac{1}{a^2-b^2} \times \frac{\left(\frac{a^2}{b^2}\right)^n - 1}{\frac{a^2}{b^2} - 1} = \frac{a^{2n}-b^{2n}}{b^{2(n-1)}(a^2-b^2)^2}.$$

11. The reciprocals of the terms being in Arithmetic Progression, let x be their common difference; then

$$x = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}.$$

Hence these reciprocals are easily found, and thence the terms of the progression.

12. See *Wood*, art. 230.

13. See *Private Tutor*, vol. i. p. 23.

14. See *Private Tutor*, Alg. pt. i. p. 200.

15. This is evident; because no part of a rational quantity can be destroyed by the opposition of sign of an irrational quantity.

16. *Wood*, art. 258.

17. Ans. $\frac{9}{2}$, 36, &c.

18. *Private Tutor*, pt. i. p. 98.

EMMANUEL COLLEGE, 1826.

1. *Wood*, arts. 38, and 40.

2. The sum $= 1l.\ 10s.\ 2d.$, and the decimal required is 7541, &c.

3. The present value

$$= \frac{210}{1+\frac{3}{4}\times\frac{6}{100}} = \frac{42000}{209} = 200l.\ 19s.\ 1d.\ 2\frac{142}{209}q.$$

See *Wood*, art. 392.

$$\therefore \text{ Gain} = 30l.\ 19s.\ 1d.\ 2\frac{142}{209}q.$$

4. $35 \text{ yards} : 32 \text{ metres} :: 1760 : \frac{1760}{35} \times 32 = 1609\frac{1}{7}.$

5. Let x feet be the breadth required; then it contains

$$36\tfrac{1}{2} \times x \text{ square feet},$$

$$\therefore \frac{5\frac{1}{4}}{20} \times \tfrac{1}{9} \times 36\tfrac{1}{2} \times x = 12 + \frac{3}{10} + \frac{1}{40} + \frac{1}{960} = 12\frac{313}{960},$$

$$\text{whence } x = 11\frac{491}{1092} \text{ feet.}$$

6. First, $.037037 \text{ \&c.} = \frac{37}{999} = \frac{1}{27}$,

$$\therefore \sqrt[3]{.037037}, \text{ \&c.} = \tfrac{1}{3} = .3333\ldots$$

$$\text{Also, } .1111\ldots = \tfrac{1}{9} = (\tfrac{1}{3})^2 = (.333\ldots)^2.$$

7. Let the number of the reinforcement $= x$; and y the quantity of each man's ration; then, $\because$

$$\text{number of men} \propto \frac{\text{food}}{\text{number of days}},$$

$$1000 + x : \frac{30000\,y - 10000\,y}{5} :: 1000 : \frac{30000\,y}{30},$$

$$\text{or } 1000 + x = \tfrac{2}{5} \times 10000 \times \tfrac{30}{3} = 4000,$$

$$\therefore\ x = 3000 \text{ men.}$$

8. The quotient is $a^4 - a^3 b + a^2 b^2 - ab^3 + b^4$.

9. Interest $\propto$ principal $\times$ rate per cent. for given time,

$\therefore$ Interest on 350 : Interest on 450 :: $350 \times 35 : 450 \times 27$,

:: 245 : 243.

10. Amount $= \mathrm{P\,R}^n$ (*Wood*, art. 397),

$\therefore$ by the question,

$$5 = \left(\frac{21}{20}\right)^n = \left(\frac{1050}{1000}\right)^n,$$

$$\therefore\ n = \frac{\log 5}{\log 1050 - \log 1000} = \frac{\log 5}{.0211893},$$

and $\log 5 = \log 10 - \log 2 = 1 - .3010300 = .69897$,

$$\therefore\ n = \frac{6989700}{211893} = 32\,\frac{209124}{211893} \text{ years.}$$

11. The number of terms is $n+2$; then, if b be the common difference,

$$1 + (n+1)\,b = 31;\ \therefore\ b = \frac{30}{n+1},$$

$$\therefore \text{ the 8th term} = 1 + 7\,b = 1 + \frac{210}{n+1} = \frac{n+211}{n+1},$$

$$\text{and } n\text{th term} = 1 + (n-1)\,b = 1 + \frac{30n-30}{n+1} = \frac{31n-29}{n+1}$$

$$\therefore\ \frac{n+211}{n+1} : \frac{31n-29}{n+1} :: 5 : 9,$$

$$\therefore\ 9\,n + 1899 = 155\,n - 145,$$

$$\text{whence } n = \frac{1022}{73} = 14.$$

12. (1). Ans. $x = 5$.

(2). Ans. $x = 5$.

(3). Ans. $x = 12$ and -2.

(4). Dividing by xy, xz, and yz, the equations become

$$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1}{a} \\ \frac{1}{x} + \frac{1}{z} &= \frac{1}{b} \\ \frac{1}{y} + \frac{1}{z} &= \frac{1}{c} \end{aligned}\right\}.$$

Adding these, and dividing

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

$$\text{Hence, } \frac{1}{x} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right),$$

$$\frac{1}{y} = \frac{1}{2}\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right),$$

$$\frac{1}{z} = \frac{1}{2}\left(-\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right),$$

$$\left.\begin{aligned} \therefore x &= \frac{2\,abc}{ac + bc - ab} \\ y &= \frac{2\,abc}{ab - ac + bc} \\ z &= \frac{2\,abc}{ab + ac - bc} \end{aligned}\right\}.$$

(5). Clearing the equation of fractions,

$$x^3 - x^{\frac{3}{2}} = 56,$$

$$\therefore x^{\frac{3}{2}} = \tfrac{1}{2} \pm \sqrt{\left(\frac{1}{4} + 56\right)} = \frac{1 \pm 15}{2} = 8 \text{ and } -7,$$

$$x = 4, \text{ and } \sqrt[3]{49}.$$

(6). Ans. $x = 3$ and $-\frac{4}{5}$.

(7). See p. 125. Ans. $x = 5$, $y = 3$.

(8). $a^{2x} - 2a^x = 8$,

$$\therefore a^x = 1 \pm 3 = 4 \text{ and } -2,$$

$$\therefore x = \frac{\log 4}{\log a} \text{ and } = \frac{\log 2}{\log a} = \frac{-\infty}{\log a}.$$

13. See *Private Tutor*, vol. i. p. 22.

14. The reason consists in this form $(\pm\sqrt{a})^2 = a$, and $\therefore$ the square root of any quantity has two values.

15. The odd combinations

$$= n + n.\ \frac{n-1}{2}\cdot\frac{n-2}{3} + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5} + \ldots$$

and the even combinations

$$= \frac{n(n-1)}{1.\ 2} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} + \ldots$$

$$\therefore \text{ odd} - \text{even} = n - \frac{n(n-1)}{1.\ 2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} - \&c.$$

$$\text{But } 0 = (1-1)^n = 1 - n + \frac{n(n-1)}{1.\ 2} - \frac{n(n-1)(n-2)}{1.\ 2.\ 3} + \&c.$$

$$\therefore \text{ odd} - \text{even} = 1.$$

16. See p. 218. No. 15.

17. Let x be the number; then

$$\frac{x-3}{4} = w \text{ an integer,}$$

$$\text{and } \frac{x-8}{9} = v,$$

$$4w + 3 = 9v + 8,$$

$$4w - 9v = 5,$$

$$\therefore w = 2v + 1 + \frac{v+1}{4}.$$

Make $v + 1 = 4u$ where u is an integer,

$$\therefore w = 9u - 1,$$

$$\therefore x = 3 + 36u - 4 = 36u - 1,$$

in which $u = 1, 2, 3$, &c.

and $\therefore x = 35, 71, 107$, &c.

18.

$$\sqrt{(14 + 8\sqrt{3})} = \sqrt{\frac{14+2}{2}} + \sqrt{\frac{14-2}{2}} = \sqrt{8} + \sqrt{6},$$

and $\sqrt{(\sqrt{8} + \sqrt{6})} = \sqrt{\frac{\sqrt{8}+\sqrt{2}}{2}} + \sqrt{\frac{\sqrt{8}-\sqrt{2}}{2}}$,

$$= \sqrt{\frac{3}{\sqrt{2}}} + \sqrt{\frac{1}{\sqrt{2}}} = \sqrt[4]{\frac{9}{2}} + \sqrt[4]{\frac{1}{2}},$$

$$= \tfrac{1}{2}(\sqrt[4]{72} + \sqrt[4]{8}).$$

Assume $\sqrt{\{4mn + 2(m^2 - n^2)\sqrt{-1}\}} = A + B\sqrt{-1}$,

$\therefore\ 4mn + 2(m^2 - n^2)\sqrt{-1} = A^2 - B^2 + 2AB\sqrt{-1}$,

$$\therefore\ A^2 - B^2 = 4\,mn,$$

$$\text{and } AB = m^2 - n^2,$$

$$\therefore\ A^2 - \frac{(m^2-n^2)^2}{A^2} = 4mn,$$

$$\therefore\ A^4 - 4mn\,A^2 = (m^2 - n^2)^2,$$

$\therefore\ A^2 - 2mn = \pm\sqrt{(2m^2n^2 + m^4 + n^4)} = \pm(m^2 + n^2)$,

$$\left.\begin{array}{l}\therefore\ A = m + n \\ \therefore\ B = m - n\end{array}\right\},$$

$\therefore$ the root required is $m + n + (m - n)\sqrt{-1}$.

This may also be resolved by the common method of extracting the root of a binomial surd.

19. Let x be the number of guineas and y that of the moidores, then $21x + 27y = 7020$,

$$\text{or } 7x + 9y = 2340,$$

for the resolution of which see p. 42. No. 21.

20. $75 = 25 \times 3 = 100 \times \frac{3}{4}$,

$\therefore\ \log 75 = 2 + \log 3 - 2\log 2$,

But $\log 3 = \frac{1}{2} \times .95424250 = .4871212$,

and $\log 2 = \frac{1}{3} \times .9030900 = .3010300$,

&c. &c.

21. It $= \dfrac{1}{3 + \dfrac{141592}{1000000}}$,

$$= \cfrac{1}{3 + \cfrac{1}{7 + \cfrac{1}{16}}} \text{ nearly},$$

$$= \cfrac{1}{3 + \cfrac{16}{113}} = \frac{113}{355} \text{ nearly.}$$

22. See *Private Tutor*, Alg. part 1. p. 98; and make $y^m = u$, then $u^n - 1$ is divisible by $u - 1$, &c. &c.

23. $(2n + 1)^2 - 1 = 4n^2 + 4n = 4n\ (n+1)$ and either n or $(n+1)$ must be even $\therefore$ &c.

24. Dividing continually by 12, the remainders will be the digits, and the numbers will be

1024, and 249.

Their product is 52710.

25. The sum of 52 cards

$$= 4\ (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) + 30,$$
$$= 4 \times 11 \times 5 + 30 = 250,$$

$\therefore$ if x be the value of the last card

$$10n + a + x = 250,$$
$$\therefore\ x = (25 - n)\ 10 - a.$$

Now x cannot exceed 10 and a is supposed less than 10,

$\therefore\ 25 - n$ cannot exceed 1 nor be less than 1,

$$\therefore\ x = 10 - a \text{ and } n = 24.$$

EMMANUEL COLLEGE, 1827.

1. *Wright's* Self-Instructions in Pure Arithmetic, p. 136.

2. $\frac{2}{7}$ of $15l. = \frac{30}{7}l.$; $3\frac{3}{7}$ of $1l. = \frac{24}{7}l.$

$\frac{1}{3}$ of $\frac{5}{7}$ of $\frac{3}{5}$ of $1l. = \frac{1}{7}l.$ and $\frac{2}{3}$ of $\frac{3}{7}$ of 1 shilling $= \frac{1}{70}l.$

$\therefore$ sum required is $\frac{30}{7} + \frac{24}{7} + \frac{1}{7} + \frac{1}{70}. = .7\frac{61}{70}l. = 7l.\ 17s.\ 5\frac{1}{7}d.$

Again, $10 = \frac{1}{2}$	1603		
$5 = \frac{1}{2}$	801	10	
$1 = \frac{1}{5}$	400	15	
$6^d = \frac{1}{2}$	80	3	
$3^d = \frac{1}{2}$	40	1	6
$1\frac{1}{2} = \frac{1}{2}$	20	0	9
	10	0	$4\frac{1}{2}$
	$1352l.$	$10s.$	$7\frac{1}{2}d.$ Ans.

3. The square root is 121. 96, &c., and the cube root is 103 exactly.

4. Interest $= 120\frac{1}{2} \times 2\frac{1}{2} \times \frac{19}{400} = 14l.\ 6s.\ 2\frac{1}{4}d.$

$$\therefore \text{ Amount } = 134l.\ 16s.\ 2\frac{1}{4}d.$$

Also,

$$\text{Interest} = 15\tfrac{1}{2} \times \text{R}^9 = \frac{31}{2} \times \left(1 \times \frac{3\frac{1}{2}}{100}\right)^9 = \frac{31}{2}\left(1 + \frac{7}{200}\right)^9;$$

$$= \frac{31}{2}\left(1 + \frac{63}{200} + 36 \times \frac{7}{40000}\right) \text{ nearly,}$$

$$= \frac{31}{2}\left(1 + \frac{63}{200} + \frac{63}{10000}\right),$$

$$= \frac{26426 \times 31}{40000} \ \&c.$$

5. Product $= x^4 - \frac{1}{2}x^4 + \frac{1}{2}x^2 + \frac{1}{2}x,$

The quotient $= px^2 + qx - r.$

6. The G. C. M. is $9a^3 b^2 (a-1)$ and the reduced fraction is

$$\frac{3a+1}{4a^2+2a-1}.$$

7. The root is $x^2 - x + \frac{1}{4}.$

8. The product is $\frac{1}{2} a^{\frac{3}{4}} - \frac{1}{3} a^{\frac{19}{30}} + \frac{1}{4} a^{\frac{1}{2}} + \frac{1}{5} a^{\frac{8}{15}} - \frac{3}{20} a^{\frac{9}{20}}$

The quotient is $a^{\frac{1}{m}-1} + a^{\frac{1}{m}-2} b + a^{\frac{1}{m}-3} b^2 + \&c.$

9. (1). First multiply by $\frac{x+1}{2}$ and reduce;

then multiply by $x + 3$.

Ans. $x = 3$.

(2). Consider $\frac{1}{x}$ and $\frac{1}{y}$ the unknown quantities; and find them by any of the common methods, &c.

$$\text{Ans. } x = \frac{mc-nb}{ac-nb};\ y = \frac{nb-mc}{b(a-m)}.$$

EMMANUEL COLLEGE, 1828.

1. See *Wright's* Pure Arithmetic.
The product is .1487992.
The quotient is .005.

2. The square root is 15367.
The cube root is 41. 1.

3. It $= 23\frac{156}{999} = 23\frac{52}{333}$.

4. (1). Interest $= \left(234 + \frac{1}{4} + \frac{1}{40}\right) \times \frac{4}{13} \times \frac{1}{20} = \&c.$

(2). Discount = future value — present value,

$$= 200 - \frac{200}{1+\frac{1}{25}} = 200\left(1 - \frac{25}{26}\right),$$

$$= \frac{100}{13} = 7l.\ 13s.\ 10d.\ \tfrac{2}{13}.$$

5. The G. C. M. is $x+6$, and the reduced fraction

$$\frac{x+5}{9x^2-x-3}.$$

6. The remainder may equal, but cannot exceed $2 \times$ root. Similarly, in extracting the cube root, the remainder may equal, but cannot exceed $3 \times$ root; and so on for higher extractions.

The most direct and simple proof of this is, perhaps, this: It is obvious, that 99, $9\dot{9}99$, $9\dot{9}999\dot{9}$, &c. leave greater remainders than any other intermediate numbers; for if 1 be added to each of them, they become perfect squares; but the roots of these are

9, 99, 999, &c.

and the remainders are

18, 198, 1998, &c.

$\therefore$ &c.

7. (1). Squaring, &c.

$$\frac{1}{a^2}+\frac{2}{ax}=\sqrt{\left(\frac{4}{a^2x^2}+\frac{9}{x^4}\right)}.$$

Squaring, &c. again,

$$\frac{1}{a^4}+\frac{4}{a^3x}=\frac{9}{x^4},$$

which cannot be solved except as a bequadratic.

But if the given equation were

$$\frac{1}{x}+\frac{1}{a}=\sqrt{\left\{\frac{1}{a^2}+\sqrt{\left(\frac{4}{a^2x^2}+\frac{9}{x^4}\right)}\right\}},$$

$$\text{then } \frac{1}{x^2}+\frac{2}{ax}=\sqrt{\left(\frac{4}{a^2x^2}+\frac{9}{x^4}\right)},$$

$$\text{and } \left(\frac{1}{x}+\frac{2}{a}\right)^2=\frac{4}{a^2}+\frac{9}{x^2},$$

$$\text{and } \frac{1}{x^2}+\frac{4}{ax}=\frac{9}{x^2},$$

$$\therefore\ x^2=2\,ax, \text{ and } x=0, \text{ and } =2\,a.$$

(2). From the second $\dfrac{x^2}{y^2}-\dfrac{16}{15}\dfrac{x}{y}=1$,

$$\therefore\ \frac{x}{y}=\frac{8\pm17}{15}=\tfrac{5}{3}, \text{ and } -\tfrac{3}{5},$$

whence $x = 3$, and 5

$y = 5$, and 3.

(3). $x = \frac{27}{5}$, and 5.

(4).
$$\begin{aligned} b^5 &= a+5\,xy\,(x^3+y^3)+10\,x^2y^2\,(x+y),\\ &= a+5\,xy\,\{b^3-3\,xy.\,(x+y)\}+10\,b\,x^2y^2,\\ &= a+5\,b^3xy-5\,bx^2y^2,\\ \therefore\ x^2y^2-b^2xy &= \frac{a-b^5}{5\,b},\\ \therefore\ xy &= \frac{b^2}{2}\pm\sqrt{\left(\frac{b^4}{4}+\frac{a-b^5}{5\,b}\right)},\\ &=\frac{b^2}{2}\pm\sqrt{\frac{4\,a-b^5}{20\,b}}, \end{aligned}$$

&c. &c.

8. Let a, b, be the extremes; then

$\dfrac{a+b}{2}$ is $>\sqrt{ab}$ is $a+b-2\sqrt{ab}$ if >0, or if $(\sqrt{a}-\sqrt{b})^2$.

$\therefore$ &c.

9. $a = \frac{x+y}{2}$, $b = \sqrt{xy}$, and $c = \frac{2xy}{x+y}$,

$$a : b :: b : \frac{b^2}{a} = \frac{xy}{\frac{x+y}{2}} = \frac{2xy}{x+y}.$$

10. See p. 233.

$$\text{Sum} = (14-138)\ 10 = -240.$$

11. (1). *Private Tutor*, vol. i. p. 21., and p. 262.

(2).

$$= a^{\frac{1}{3}}\left(1 - \frac{1}{3} \cdot \frac{x}{a} - \frac{1}{9}\frac{x^2}{a^2} - \frac{5}{81}\frac{x^3}{a^3} - \frac{10}{243}\frac{x^4}{a^4} - \&c.\right)$$

12. $2^n = (1+1)^n = 1 + n + \frac{n(n-1)}{1.\ 2} + \&c.$

13. See p. 235. No. 15.

14. The number $= \dfrac{n\,(n-1)\,(n-2)\ldots 3.\ 2.\ 1}{1.\,2.\,3\ldots p \times 1.\,2.\,3\ldots q \times 1.\,2.\,3\ldots r\ \&c.}$

For, if x be the number required; then, it is clear, if the p things, q things, r things, &c. become all different, instead of the same in groups, that the whole number of permutations is

$$P \times 1.\ 2.\ 3 \ldots p \times 1.\ 2\ 3 \ldots q \times 1.\ 2.\ 3 \ldots r, \&c.$$

But this number is also $1.\ 2.\ 3\ \ldots\ (n-2)\,(n-1)\,n$,

$$\therefore P = \frac{1.\ 2.\ 3\ \ldots\ (n-2)\,(n-1)\,n}{1.\,2.\,3\ldots p \times 1.\,2.\,3\ldots q \times 1.\,2.\,3\ldots r \times \&c.}.$$

In the word *Constantinople*, there being three n's, two o's, and two t's; the number required is (in this case)

$$\frac{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10.\ 11.\ 12.\ 13}{1.\ 2.\ 3 \times 1.\ 2 \times 1.\ 2},$$

$$= 5.\ 6.\ 7.\ 8.\ 9.\ 10.\ 11.\ 12.\ 13,$$

$$= 15120 \times 17160 = \&c.$$

15. *Private Tutor*, Alg. pt. i. p. 98.

16. The root is $2 - \sqrt{3}.\ \sqrt{-1}$.

17. The numbers being in Arithmetic Progression, let them be $x-y$, x, $x+y$; then

$$\left.\begin{aligned} x-y \;+x\;+\;x+y &= 15 \\ \text{and } (x-y)^3+x^3+(x+y)^3 &= 495 \end{aligned}\right\},$$

$$\therefore\; x = 5, \text{ and } y = \tfrac{4}{5},$$

$\therefore$ the numbers are

$$\tfrac{21}{5},\; 5,\; \tfrac{29}{5}.$$

18. For, in that system, if d, d', d'', &c. be the digits of any number N, we have hence

$$\begin{aligned} \text{N} &= d+d'.\,a+d''.\,a^2 + \text{\&c.} \\ &= d+d'\,(a-1+1)=d''\,(a-1+1)^2 + \text{\&c.} \\ &= d+d'+d''+\text{\&c.}+(a-1)\times(d'+\text{\&c.}) \\ &\text{\&c.} \end{aligned}$$

CHRIST'S COLLEGE, 1824.

1. Here is a misprint; for $12\,p+p$ read $12\,p+q$,

$$\frac{m}{n} = \frac{12p+q}{240} = \frac{p}{20} + \frac{q}{240} = p \text{ shillings} + q \text{ pence.}$$

2. It $= \dfrac{2.2}{200} = .11.$

3. $.1111\ldots\ldots = \frac{1}{9} = (\frac{1}{3})^2 = (.3333\ldots)^2.$

4. Reduce them to the same denominator, &c.

5. It $= t - \dfrac{m}{n} - \dfrac{p}{q} = \dfrac{nqt-mq-np}{nq}.$

6. Instead of $(ba > b)$ it ought to be $b\,(a > b)$, and the (h) should be (p).

The form is true; for if u, v, be the remainders,

$$\begin{aligned} a &= bp+u, \\ b &= uq+v. \\ u &= vr, \end{aligned}$$

$$b = uq + \frac{u}{r} = \frac{qr+1}{r}.\ u,$$

$$\therefore u = \frac{br}{qr+1},$$

$$\therefore a = bp + \frac{b}{qr+1} = b.\ \frac{pqr+1}{r}.$$

7. The G. C. M. is $x-1$; and the reduced fraction is

$$\frac{5x^2-3x-2}{5x-7}.$$

8. For $r = \frac{q^2}{3p} = \left(\frac{p^2}{3}\right)^2 \div 3p = \frac{p^3}{27}$,

$$\therefore 1 + px + \frac{p^2}{3}x^2 + \frac{p^3}{27}x^3 = \left(1 + \frac{px}{3}\right)^3.$$

9. *Wood*, art. 380.

10. The means are $\frac{a+b}{2}$, $\sqrt{ab}$, $\frac{2ab}{a+b}$.

Let $a = b + \delta$, δ being a small quantity,

then $\frac{a+b}{2} = \frac{2b+\delta}{2}$,

$$\sqrt{ab} = \sqrt{b}\,(b+\delta)^{\frac{1}{2}} = b\left(1 + \frac{\delta}{b}\right)^{\frac{1}{2}} = b\left(1 + \frac{\delta}{2b}\right) \text{ nearly},$$

$$= \frac{2b+\delta}{2}.$$

$$\text{and } \frac{2ab}{a+b} = \frac{2(b+\delta)b}{2b+\delta} = 2b(b+\delta)(2b+\delta)^{-1},$$

$$= (b+\delta)\left(1 + \frac{\delta}{2b}\right)^{-1},$$

$$= (b+\delta)\left(1 - \frac{\delta}{2b}\right) \text{ nearly},$$

$$= \frac{(2b-\delta)(b+\delta)}{2b} = \frac{2b^2+2b\delta-b\delta}{2b} \text{ nearly}$$

$$= \frac{2b+\delta}{2} \text{ nearly},$$

$\therefore$ &c.

11. The sum $= \frac{f}{g} - \frac{f-g}{g^2} x . \left(1 - \frac{x}{g} + \frac{x^2}{g^2} - \&c.\right),$

$$= \frac{f}{g} - \frac{f-g}{g^2} x . \frac{1}{1+\frac{x}{g}} = \frac{f}{g} - \frac{f-g}{g} . \frac{x}{g+x},$$

$$= \frac{f+x}{g+x}.$$

Observe. It ought to have been stated that the series is convergent.

12. *Wood*, art. 209.

13. *Wood*, art. 102.

14. Let $S = \sqrt{(a+b\sqrt{-1})} + \sqrt{(a-b\sqrt{-1})}$;

then (b is omitted in the first term),

$$S^2 = 2a + 2\sqrt{(a^2+b^2)},$$

$$\therefore S = \sqrt{\{2a+2\sqrt{(a^2+b^2)}\}}.$$

15. (α). $x = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}$.

(β). This should be $\sqrt{(x+9)} = 1 + \sqrt{x}$.

Ans. $x = 16$.

(γ). Ans. $x = \frac{abc\,(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$, $y = \frac{abc\,(ac-ab-ac)}{a^2b^2+a^2c^2-b^2c^2}$.

(δ). The answer is $x = \frac{5}{3}$ and $-\frac{3}{2}$.

(ϵ). First $\frac{a}{\sqrt{(a+x)}} = \frac{b}{x} - \frac{x}{a+x}$.

Squaring, &c. $\frac{x^4}{(a+x)^2} - (a^2+2b) . \frac{x^2}{a+x} = -b^2,$

$$\therefore \frac{x^2}{a+x} = \frac{a^2+2b \pm a\sqrt{(a^2+4b)}}{2} = 2A \text{ suppose},$$

$$\therefore x^2 - 2Ax = 2aA,$$

whence the *four* values of x.

(η). First $x^3 - x = 2(x+1)$,

$$\therefore x(x+1)(x-1) = 2(x+1),$$

$$\therefore (x+1)(x^2-x-2) = 0,$$

$$\therefore\ x = -1, \text{ and } x^2 - x = 2,$$
$$\therefore\ x = \tfrac{1}{2} \pm \tfrac{3}{2} = 2, \text{ and } x = -1,$$
$$\therefore\ \text{the roots are } -1,\ -1, \text{ and } 2.$$

16. For, by means of the $(n+p)$ equations, only p of the unknown quantities can be eliminated so as to leave n independent equations.

17. If the roots be possible, they are both negative when all the three terms have the same sign, and both positive, when the three terms are not all alike.

The roots are both impossible, when the last term of $x^2 + p\,x + q = 0$ is positive, and $p^2 < 4\,q$.

18. Let x be the distance from A at which they meet, and y the number of days in doing it; then

$$\left.\begin{aligned} a\,y &= x \\ by &= \mathrm{D} - x \end{aligned}\right\},$$

$$\therefore\ y = \frac{\mathrm{D}}{a+b}, \text{ and } x = \frac{a\,\mathrm{D}}{a+b}.$$

a and b would have different signs if A or B were to run after the other; that is, if they both went in the same direction.

The negative sign is merely introduced by the operation, the velocities a and b having no reference to direction.

19. Let x be the first term, y the number of terms; then

$$x + (y-1) \times 2 = 19,$$

$$\text{and } \{2x + (y-1) \times 2\}\,\frac{y}{2} = 91,$$

$$\therefore\ \left.\begin{aligned} (19+x)\,y &= 182 \\ x + 2\,y &= 21 \end{aligned}\right\},$$

$$\left.\begin{aligned} \text{whence } x &= -5 \text{ and } 7 \\ y &= 13 \text{ and } 7 \end{aligned}\right\} \text{ which explain themselves.}$$

They are easily verified.

20. Let x, y, $2x$ be the digits; then

$$\left.\begin{aligned}\frac{2x+10y+100x}{y+3x}&=22\\ \frac{102x+10y}{xy}&=11\end{aligned}\right\},$$

$$\left.\begin{aligned}51x+5y&=11y+33x\\ 102x+10y&=11xy\end{aligned}\right\},$$

$$\left.\begin{aligned}\therefore\ x&=4\\ y&=12\end{aligned}\right\},$$

which give the answer, although 12 is too large for a digit.

21. See p. 219. No. 17.

22. See *Barlow's* Theory of Numbers.

23. *Wood*, art. 405.

24. The amount $= \mathrm{P\,R}^n$, (*Wood*, art. 397.)

$$\therefore\ 3\mathrm{P} = \mathrm{P\,R}^n, \text{ or } \mathrm{R}^n = \left(1+\frac{3\frac{1}{2}}{100}\right)^n = \left(\frac{207}{200}\right)^n = 3,$$

$$\therefore\ (1.035)^n = 3,$$

$$\therefore\ n = \frac{\log 3}{\log 1.035} = \frac{4771213}{149403} = 32\frac{317}{149403} \text{ years.}$$

25. This question is erroneously stated; for the first is the only term of the expansion that can involve a^n, and no part of the expansion (at least according to the Polynomial Theorem) of $(a+b+c)^n$, can be negative.

26. The number is $\dfrac{1.\,2.\,3.\ldots\ldots..(m+n-1)\,(m+n)}{1.\,2.\,3.\ldots\ldots..m\times 1.\,2.\,3.\ldots..n}$, see p. 241.

27. Were the judgments not supposed already given, the probability that they would agree in awarding a just decision, would be

$$\frac{a}{a+b}\times\frac{c}{c+d}, \text{ or } \frac{ac}{(a+b)(c+d)}.$$

The probability of their agreeing in an unjust decision would be

$$\frac{bd}{(a+b)(c+d)}.$$

But after it is known that their decisions accord, the probability that both are just, is

$$\frac{ac}{ac+bd},$$

and the probability that both are erroneous, is

$$\frac{bd}{ac+bd}.$$

Also, the probability that the first delivers a just, and the other an erroneous judgment, is

$$\frac{ad}{ad+bc},$$

and the probability that the second is just, and the first the contrary, is

$$\frac{bc}{ad+bc}.$$

CHRIST'S COLLEGE, 1828.

1. $\left.\begin{array}{l}\frac{7}{9} \text{ of a pound } = \frac{140}{9} \text{ shillings} \\ \frac{2}{5} \text{ of a guinea } = \frac{42}{5} \text{ shillings}\end{array}\right\}$, the difference of which is exactly $\frac{322}{45}$ shillings $= 7s.\ 1d.\ 3\frac{7}{15}$ farthings.

2. The sum is 3.125.

3. The second quantity in the original paper is

$$x^4 - x^2y - x^2y^2 + y^3.$$

The G. C. M. is $x^2 - y^2$.

4. The square root is $\dfrac{x}{y} + \dfrac{y}{x} - \dfrac{1}{\sqrt{2}}$.

$$(1-x)^{\frac{1}{3}} = 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \text{\&c.}$$

5. *Wood*, art. 377.

6. See p. 33. No. 2.

7. Let x, y, be the numbers; then

$$\left.\begin{aligned} x+y &= xy \\ x+y &= x^2-y^2 \end{aligned}\right\} \therefore \quad \begin{aligned} x &= \frac{3\pm\sqrt{5}}{2} \\ y &= \frac{1\pm\sqrt{5}}{2}. \end{aligned}$$

8. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}} = \dfrac{(8-5\sqrt{2})(3+2\sqrt{2})}{9-8} = 4+\sqrt{2}.$

$$\sqrt{-x}.\sqrt{-y} = -\sqrt{(xy)}, \sqrt[4]{-x}\times\sqrt[4]{-y} = \sqrt{\{-\sqrt{(xy)}\}}$$
$$= \sqrt[4]{(xy)}.\sqrt{-1},$$
$$\sqrt[6]{}=x\times\sqrt[6]{-y} = \sqrt{(\sqrt[3]{-x})}\times\sqrt{(\sqrt[3]{-y})} = \sqrt{(-\sqrt[3]{x})}$$
$$\times\sqrt{(-\sqrt[3]{y})} = \sqrt{-1}.\sqrt{-1}.\sqrt[6]{(xy)}. = -\sqrt[6]{(xy)}.$$

9. *Wood*, art. 162.

10. *Wood*, art. 252. This is imperfectly expressed in *Wood*. It aught to have been; the square root of a *rational quantity*, &c.

11. *Wood*, art. 222.

The fraction $= \frac{6}{9} = \frac{2}{3}$,

12. *Wood*, art. 182.

13. *Wood*, art. 220.

14. *Wood*, art. 189.

15. If possible let $\left.\begin{aligned} a+b &= mp \\ \text{and } a-b &= mq \end{aligned}\right\}$ m being their common measure,

then $2a = m(p+q)$ and $2b = m(p-q)$,

$\therefore$ $2a$ and $2b$ have a common measure m, but they can have no common measure but 2, for a and b are prime to each other; $\therefore$ a $+ b$ and $a-b$ can have no common measure but 2.

16. For all numbers, prime or composite, must be of one of the forms

$$4m,\ 4m+1,\ 4m+2,\ 4m+3;$$

but prime numbers, greater than two, cannot belong to $4m$ and $4m+2$,

$\therefore$ all primes must be included in the forms

$$4m+1 \text{ and } 4m+3,$$
$$\text{or, } 4m+1 \text{ and } 4(m+1)-1;$$
$$\text{that is, } 4m+1 \text{ and } 4m-1.$$

17. The series will never terminate excepting when n is a positive integer; and then the $(n+1)^{\text{th}}$ term is the last. (See *Private Tutor*, Alg. pt. i. p. 199.)

18. *Wood*, art. 380.

19. *Wood*, art. 285.

$$\text{Also, } \sqrt{-49} = 7\sqrt{-1}.$$

This would seem to have been a mistake of the original paper, and we suspect instead of -49 it should have been $\sqrt{-49}$. This being presumed,

$$\sqrt{(0+\sqrt{-49})} = \sqrt{\frac{7+0}{2}} + \sqrt{\frac{7-0}{2}}\cdot\sqrt{-1},$$
$$= \sqrt{\frac{7}{2}} + \sqrt{\frac{7}{2}}\cdot\sqrt{-1}.$$

20. The first series is recurring; its scale of relation is $f+g+h+k = 4-6+4-1$; and the sum is

$$\frac{1+6x+x^2}{(1-x)^4}.$$

The second series may be summed by Subtraction, Multiplication, or by the Method of Differences.

$$\text{Its sum is } \frac{n}{n+1}.$$

21. See *Wood*, art. 228, and p. 241.

22. See p. 196.

23. (1). $x = \dfrac{\sqrt{a}}{\sqrt{a+2}}$.

(2). $x + y + 2\sqrt{(xy)} = 36,$

$$x + y = 20,$$

$$\therefore 2\sqrt{(xy)} = 16.$$

Hence $x = 16$ and 4, $y = 4$ and 16.

(3). $x^2 - 16 + \sqrt{(x^2-16)} = 12;$

whence $x^2-16 = 9$ and 16.

$\therefore x = \pm 5$ and $\pm 4\sqrt{2}$.

(4). Let $\dfrac{a}{b}$, a, ab, be the roots; then

$$a^3 = 27, \therefore a = 3,$$

$$\therefore \frac{3}{b} + 3 + 3b = 13,$$

$$b^2 - \tfrac{10}{3}b = 1,$$

$$b = \frac{5}{3} \pm \sqrt{\frac{16}{9}} = \frac{5 \pm 4}{3} = 3 \text{ and } \frac{1}{3},$$

$\therefore$ the roots are

$$1, \quad 3, \quad 9.$$

ST. PETER'S COLLEGE, 1827.

1. The square root is 203975.

First, $\frac{1}{17} = .058823529$,

the root of which is .2425, &c.

2. In four years the Amount would be

$$760\tfrac{1}{2}.\ (1 + \tfrac{1}{25})^4\ (M = PR^n),$$

$$\text{or, } \frac{1521}{2} \times \frac{26^4}{25^4},$$

$$\therefore \text{the Interest} = \frac{1521}{2}\left(\frac{26^4}{25^4} - 1\right),$$

$$= \frac{1521}{2} \times \frac{26^4 - 25^4}{25^4},$$

$$= \frac{1521}{2} \times \frac{66351}{390625},$$

which is easily reduced to pounds, shillings, &c.

3. $\frac{4}{7}$ cwt. $= \frac{16}{7}$ qrs. $= 2\frac{2}{7}$ qrs. $= 2$ qrs. 8 lbs.

Also $\frac{5}{6}$ lb. $= \frac{80}{6}$ oz. $= 13\frac{1}{3}$ oz.

Hence the sum stands

qrs.	lb.	oz.
2 .	8 .	0
	8 .	$13\frac{1}{3}$
		$3\frac{9}{10}$
2 .	17 .	$1\frac{7}{30}$.

4. *Wright's* Pure Arithmetic.

5. *Wood*, for Rule,

$$\frac{42237}{75582} = \frac{19}{34}.$$

Also, $\dfrac{8a^2b^2-10ab^3+2b^4}{9a^4b-9a^3b^2+3a^2b^3-3ab^4} = \dfrac{2b}{3a} \cdot \dfrac{4a+b}{3a^2+b^2}.$

6. (1). First multiply by 12, &c.

$$\text{Ans. } x = -\frac{8785}{117}.$$

(2). Ans. $\left.\begin{array}{l} x = 5 \\ y = 5 \end{array}\right\}.$

(3). By reduction, we get

$$z^2 - \frac{2191}{226}z = -\frac{4065}{226},$$

whence z by the solution of the quadratic.

(4). From the first,

$$xy = -3 \pm 5 = 2, \text{ and } -8.$$

From the second,

$$y = x+1,$$

$$\therefore x^2+x = 2, \text{ and } -8,$$

$$\therefore x = -\tfrac{1}{2} \pm \tfrac{3}{2} = 1 \text{ and } -2;$$

$$\text{also} = -\tfrac{1}{2} \pm \frac{\sqrt{-31}}{2},$$

$$\therefore y = 1, -1, \tfrac{1}{2} \pm \frac{\sqrt{-31}}{2}.$$

(5). The first gives, by reduction,

$$x^4 - 8\sqrt{y}.\ x^2 = 9y,$$

$$\therefore\ x^2 = 4\sqrt{y} \pm 5\sqrt{y} = 9\sqrt{y}, \text{ and } -\sqrt{y}\ \ldots\ (1).$$

The second gives, by reduction,

$$y^3 + x^2 y^{\frac{3}{2}} = \frac{10}{81}x^4,$$

$$\therefore\ y^{\frac{3}{2}} = -\frac{x^2}{2} \pm \frac{11}{2}x^2 = 5x^2, \text{ and } -6x^2\ \ldots\ (2).$$

Hence, $y^{\frac{3}{2}} = 5 \times 9\sqrt{y}$, and $-6 \times 9\sqrt{y}$,

$\therefore\ y = 45$, and -54.

Also, $y^{\frac{3}{2}} = 5 \times -\sqrt{y}$, and $-6 \times -\sqrt{y}$,

$\therefore\ y = -5$ and 6;

whence the corresponding values of x^2 are

$$9\sqrt{45},\ 9\sqrt{-54};$$
$$\text{and } -\sqrt{-5},\ -\sqrt{6};$$
$$\text{or, } 27\sqrt{5},\ 27\sqrt{-6};$$
$$\text{and } -\sqrt{-5},\ -\sqrt{6}.$$

$\therefore$ the simultaneous values are

$$x = \pm\sqrt{(27\sqrt{5})}, \pm\sqrt{(27\sqrt{-6})}, \pm\sqrt{(-\sqrt{-5})}, \pm\sqrt{(-\sqrt{6})},$$
$$y = 45,\qquad -54,\qquad -5,\qquad 6,$$

of which the imaginary and surd values admit further reduction.

7. For a proof as general, and far easier of comprehension, see *Private Tutor*, vol. i. p. 22. Those who wish to see *Euler's* Demonstration, may consult tom. xix. Novi Comment. Acad. Petrop. p. 103; or *Lacroix's* Complément des Elemens d' Alegebra, p. 159.

8. See p. 46. No. 9.

9. (1). Sum $= (2 + 99 \times 8)\ 50 = 39700$.

(2). Sum $= (30 - 15 \times \frac{1}{3})\ 8 = 200$.

(3). $S = \dfrac{\frac{1}{4}}{1 + \frac{1}{4}} = \frac{1}{5}$.

(4).

$$S = x^{\frac{5}{2}} \frac{\left(\frac{-a}{x\sqrt{x}}\right)^n - 1}{\left(\frac{-a}{x\sqrt{x}}\right) - 1} = \frac{x^{\frac{5}{2}}}{(x\sqrt{x})^{n-1}} \cdot \frac{(x\sqrt{x})^n - (-a)^n}{a + x\sqrt{x}},$$

$$= x^{\frac{-3n-8}{2}} \cdot \frac{(x\sqrt{x})^n - (-a)^n}{a + x\sqrt{x}}.$$

10. *Private Tutor*, vol. i. p. 200.

11. $\sqrt{(2+\sqrt{3})} = \sqrt{\tfrac{3}{2}} + \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{6} + \tfrac{1}{2}\sqrt{2}.$

$$\sqrt{(3\sqrt{6}+2\sqrt{12})} = \sqrt{\frac{3\sqrt{6}+\sqrt{6}}{2}} + \sqrt{\frac{3\sqrt{6}-\sqrt{6}}{2}}$$

$$= \sqrt{(2\sqrt{6})} + \sqrt{(\sqrt{6})},$$

$$= \sqrt[4]{24} + \sqrt[4]{6}.$$

$$\sqrt{\{xy - 2x\sqrt{(xy-x^2)}\}} = \sqrt{x}.\sqrt{\{y - 2\sqrt{(xy-x^2)}\}},$$

$$= \sqrt{x}.\left\{\sqrt{\frac{y+y-2x}{2}} - \sqrt{\frac{y-y+2x}{2}}\right\}$$

$$= \sqrt{x}.\{\sqrt{(y-x)} - \sqrt{x}\}.$$

12. Let x be the number of the 21*s.* 6*d.* coins, and y that of the other coins; then

$$43x + 34y = 4000,$$

$$\therefore y = 117 - x + \frac{22-9x}{34}.$$

Make $22 - 9x = 34w$,

$$\therefore x = 2 - 3w + \frac{4-7w}{9}.$$

Make $4 - 7w = 9v$,

$$\therefore w = -v + \frac{4-2v}{7}.$$

Make $4 - 2v = 7u$,

$$\therefore v = 2 - 3u - \frac{u}{2}.$$

Make $\frac{u}{2} = t$,

$$\therefore v = 2 - 7t,$$

$$\therefore w = 9t - 2,$$

$$\left.\begin{aligned}\therefore x &= 10-34t\\ \therefore y &= 15+43t\end{aligned}\right\}.$$

Hence, it appears, that 10 coins of the first kind, and 15 of the second, give the only solution.

13. $(5x^2y^2z-4xz^2)^4 = 625x^8y^8z^4\left(1-\dfrac{4z}{5xy^2}\right)^4,$

$$= 625x^8y^8z^4\left(1-\frac{16z}{5xy^2}+\frac{96z^2}{25x^2y^4}-\frac{256z^3}{125x^3y^6}+\frac{256z^4}{625x^4y^8}\right),$$

$$= 625x^8y^8z^4-2000x^7y^6z^5+2400x^6y^4z^6-1280x^5y^2z^7$$
$$+256x^4z^8.$$

$$\sqrt[3]{(1-x)} = (1-x)^{\frac{1}{3}} = 1-\frac{x}{3}-\frac{x^2}{9}-\frac{5x^3}{81}-\frac{10x^4}{243}-\text{\&c.}$$

$$(1+x^2)^{-3} = 1-3x^2+6x^4-10x^6+\text{\&c.}$$

14. The present worth of P pounds to commence after p years, and continue thence to q years = its P value to continue from the first for $(p+q)$ years—its P value from the first for p years

$$= \frac{1-\dfrac{1}{R^{p+q}}}{R-1}.A-\frac{1-\dfrac{1}{R^p}}{R-1}A = \frac{A}{R^p}.\frac{1-\dfrac{1}{R^q}}{R-1}.$$

When q is infinite this expression becomes

$$\frac{A}{R^p(R-1)},$$

and the required present value $= \dfrac{100}{R^p(R-1)}.$

15. The cost = 3*l*. 1*s*. 5*d*.

ST. PETER'S COLLEGE, 1828.

1. The approximate fraction required is $\frac{530}{69}$.

2. The general form of Triangular Numbers is $\dfrac{n(n+1)}{1.\ 2}$, whilst that of Pentagonal Numbers is $\dfrac{n(3n-1)}{1.\ 2}$.

Hence, to resolve the question, let the x^{th} triangular number $= y^{\text{th}}$ pentagonal number; then

$$x(x+1) = y(3y-1),$$
$$\text{or, } x^2+x = 3y^2-y,$$
$$\therefore\ x^2-y^2+x+y = 2y^2,$$
$$\therefore\ x+y+1 = \frac{2y^2}{x+y},$$

$\therefore\ \dfrac{2y^2}{x+y}$ must be an integer; which can only be when

$$x = y. \quad \text{But, then,}$$
$$\frac{x(x+1)}{1.\,2} = \frac{x(3x-1)}{1.\,2},$$
$$\therefore\ x^2+x = 3x^2-x,$$
$$x^2 = x, \text{ or } x = 1,$$

$\therefore$ 1 is the only triangular number which is a pentagonal number.

3. First,

$$fa_1+fa_3+\ldots fa_{2n-1} = \tfrac{1}{2}\{u^a+u^{3a}+\ldots u^{(2n-1)a}\}$$
$$-\tfrac{1}{2}\{u^{-a}+u^{-3a}+\ldots\ldots u^{-(2n-1)a}\},$$
$$= \tfrac{1}{2}\,u^a.\frac{(u^{2a})^n-1}{u^{2a}-1} - \tfrac{1}{2}\,u^{-a}.\frac{(u^{-2a})^n-1}{u^{-2a}-1},$$
$$= \tfrac{1}{2}\,\frac{u^a}{u^{2a}-1}\{(u^{2a})^n+)u^{-2a})^n-2\},$$
$$\therefore\ fa_1(fa_1+fa_3+\ldots fa_{2n-1}) = \tfrac{1}{4}(u^{2an}+u^{-2an}-2)$$
$$= \left(\frac{u^{an}-u^{-an}}{2}\right)^2 = (fa_n)^2.$$

4. Let x be the first term, y the common ratio; then, if s be the sum of the odd terms, and s' that of the even terms, we have

$$s = x.\ \frac{y^n-1}{y-1},\ s' = xy^n.\ \frac{y^n-1}{y-1},$$
$$\therefore\ y = \left(\frac{s'}{s}\right)^{\frac{1}{n}},$$

$$\text{Also, } x = s.\,\frac{y-1}{y^n-1} = s.\ \frac{\left(\frac{s'}{s}\right)^{\frac{1}{n}}-1}{\frac{s'}{s}-1} = \frac{s'^{\frac{1}{n}}-s^{\frac{1}{n}}}{s'-s}.\ s^{2-\frac{1}{n}},$$

$\therefore$ the p^{th} term $= xy^{p-1} = \dfrac{s'^{\frac{1}{n}} - s^{\frac{1}{n}}}{s'-s} \cdot s'^{\frac{p-1}{n}} \cdot s^{2-\frac{p}{n}} = \text{P}$ suppose,

and the q^{th} term $= xy^{q-1} = \dfrac{s'^{\frac{1}{n}} - s^{\frac{1}{n}}}{s'-s} s'^{\frac{q-1}{n}} \cdot s^{2-\frac{q}{n}} = \text{Q}$ suppose.

Again, let z be the Common Difference of the terms of the Arithmetic Series of which P and Q are the extremes; then

$$\text{P} + (r+1)\, z = \text{Q},$$

$$\therefore z = \frac{\text{Q}-\text{P}}{r+1},$$

$$\text{and the } r^{\text{th}} \text{ mean} = \text{Q} - \frac{\text{Q}-\text{P}}{r+1} = \frac{r\text{Q}+\text{P}}{r+1},$$

$$= \frac{1}{r+1} \cdot \frac{s'^{\frac{1}{n}} - s^{\frac{1}{n}}}{s'-s} \cdot \frac{s^2}{s'^{\frac{1}{n}}} \cdot \left\{ r\left(\frac{s'}{s}\right)^{\frac{q}{n}} + \left(\frac{s'}{s}\right)^{\frac{p}{n}} \right\}.$$

5. See *Private Tutor*, Alg. pt. i. p. 260. Ex. 41.

6. In this y_5 should be y_3 and *vice versa*.

Make $\text{S} = \dfrac{1}{\sqrt{3}} + \dfrac{1}{1+\sqrt{3}} + \dfrac{1}{2+\sqrt{3}} + \ldots\ldots \infty,$

$$\therefore \text{S} - \frac{1}{\sqrt{3}} = \frac{1}{1+\sqrt{3}} + \frac{1}{2+\sqrt{3}} + \frac{1}{3+\sqrt{3}} + \ldots \infty,$$

$$\therefore \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}(1+\sqrt{3})} + \frac{1}{(1+\sqrt{3})(2+\sqrt{3})} + \ldots \infty,$$

$$\therefore y_3 = \frac{1}{\sqrt{3}}.$$

Similarly,

$$y_5 = \frac{1}{\sqrt{5}},\ y_7 = \frac{1}{\sqrt{7}},\ \ldots\ldots\ y_{2n-1} = \frac{1}{\sqrt{(2n-1)}},$$

$$\therefore \frac{1}{(y_3)^2} + \frac{1}{(y_5)^2} + \ldots (n-1) \text{ terms} = \frac{1}{3} + \frac{1}{5} + \ldots \frac{1}{2n-1},$$

which cannot be generally summed.

7. This question properly belongs to the Theory of Symmetrical Functions. See *Private Tutor*, vol. i. p. 271.

It may easily be proved, by showing it to be true for any simple case, such as the squares; then assuming it for a general index p; and proving it true for the index $p+1$, provided that assumption be true, and so on, as will be evident.

8. Since,

$$p_3-p_2 = n(n-1)(n-2)-n(n-1) = n(n-1)\times(n-3),$$

$$p_4-p_3 = \ldots\ldots\ldots\ldots = n(n-1)(n-2)\times(n-4),$$

$$\&c. = \&c.$$

$$p_{n-1}-p_{n-2} = n(n-1)(n-2)\ldots\ldots 4.\ 3\times 1,$$

$$\therefore (p_3-p_2)(p_4-p_3)\ldots(p_{n-1}-p_{n-2}) = n(n-1)\times n(n-1)$$

$$+(n-2)\times\ldots n(n-1)\ldots 4.3\times(n-3)(n-4)(n-5)\ldots 3.2.1.$$

$$= p_2\, p_3\ldots p_{n-2}\times\frac{p_n}{p_3} = \frac{p_2\, p_3\ldots\ldots p_{n-2}\, p_{n-1}\, p_n}{p_3\, p_{n-1}},$$

$$= \frac{\mathrm{P}}{p_3\, p_{n-1}},$$

$$\therefore \mathrm{P} = p_3\, p_{n-1}(p_3-p_2)(p_4-p_3)(p_5-p_4)\ldots(p_{n-1}-p_{n-2}).$$

9. First, (*Wood*, art. 224)

$$s_1 = \frac{a}{1-r},\quad s_2 = \frac{a^2}{1-2r},\quad s_3 = \frac{a^3}{1-3r},\ \&c.$$

$$\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}+\ldots\infty = \frac{1-r}{a}+\frac{1-2r}{a^2}+\frac{1-3r}{a^3}+\ldots\infty.$$

$$\text{Let } \mathrm{S} = \frac{1-r}{a}+\frac{1-2r}{a^2}+\&c.$$

$$= \frac{1-r}{a}-\frac{r}{a^2}-\frac{r}{a^3}-\&c.$$

$$+\frac{1-r}{a^2}+\frac{1-2r}{a^3}+\frac{1-3r}{a^4}+\&c.$$

$$= \frac{1-r}{a}-\frac{r}{a^2}.\frac{1}{1-\frac{1}{a}}+\frac{1}{a}.\ \mathrm{S},$$

$$\therefore \mathrm{S}.\frac{a-1}{a} = \frac{a(1-r)-1}{a(a-1)},$$

$$\therefore \mathrm{S} = \frac{a(1-r)-1}{(a-1)^2}.$$

10. The general values of x and y are

$$x = 269 - 19w,$$
$$y = 2 + 7w,$$

w being any whole number.

$$\therefore\ x+y = 271-12w,$$

in which, if $w = 22$,

$$x+y = 7,$$

which is the least positive value.

This value will not give the series of the question; but, if the least value of $x+y$ were 103, which would correspond to $w = 14$, then we should have

$$103 = \frac{3}{1-\frac{100}{103}} = \frac{3}{1-\frac{1}{1.03}},$$

$$\text{and } 103^n = 3^n\left(1-\frac{1}{1.03}\right)^n,$$

$$= 3^n\left\{1+n.\frac{1}{1.03}+\frac{n(n+1)}{1.\ 2}.\left(\frac{1}{1.03}\right)^2+\&c.\right\}.$$

It is manifest 103 is not the least sum of x and y which will satisfy the equation $7x+19y = 1921$.

It must be obvious to every one, in fact, that this paper is not only erroneous, but exceedingly injudicious. It is much beyond the experience of those for whom it was designed.

ST. PETER'S COLLEGE, 1830.

1. (1). Ans. $195l.\ 11s.\ 7\frac{1}{28}d.$

(2). $\frac{3}{5}$ of $\frac{8}{13}$ of a moidore $= \frac{24}{65}\times 27$ shillings,

and $2\frac{7}{9}$ of $8s.\ 5d. = \frac{25}{9}\times\frac{101}{12}$ shillings,

$$\therefore\ \text{sum} = \frac{24\times 27}{65}+\frac{25\times 101}{108}+\frac{100}{9}\text{ shillings},$$
$$= 2l.\ 4s.\ 5\tfrac{1}{2}\tfrac{2}{45}d.$$

2. *Wood*, art. 93.

The G. C. M. of 36.595 and 5.7980 is 65.

The other G. C. M. is $a\,(2ax-3y^2)$, and the reduced fraction

$$\text{is } \frac{x^2\,(3a^2-5xy)}{y^2\,(5a^2+4xy)}.$$

3. $$.02777\ldots\ldots = \frac{2}{100} + \frac{7}{9000} = \frac{187}{9000},$$

$$\text{and } \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{3}{4}\ldots = \tfrac{1}{3}\cdot\frac{1}{1+\frac{3}{2}} = \frac{2}{15},$$

$$\therefore \text{ the reduced fraction} = \frac{187}{9000}\times\frac{15}{2} = \frac{187}{1200}.$$

4. (1). The root is $x^2+x-\frac{1}{2}$.

(2). The root of the other quantity is

$$\sqrt{(-a^{\frac{2}{m}}.\ b^{\frac{2}{n}}.\ c^{-r})} + \sqrt{(-a^{\frac{2}{n}}.\ c^{\frac{2}{n}}.\ b^{-r})}.$$

5.
```
2 t t 9) 2 9 t 9 6 5 8 0 (e 7 t 7
         2 7 e t 3
         ---------
           1 t e 2 5
           1 8 4 3 3
           ---------
             2 6 e 2 8
             2 5 0 e 6
             ---------
               1 t 3 2 0
               1 8 4 3 3
               ---------
                 1 t t 9
```

Again, let r be the radix of the required system; then

$$9+9r^2+9r^4 = 22059,$$

$$\therefore r^4+r^2 = 2451-1 = 2450,$$

$$\therefore r^2 = -\tfrac{1}{2} \pm \sqrt{\frac{9801}{4}} = \frac{-1+99}{2} = 49,$$

$$\therefore r = 7.$$

This question is imperfectly stated, the digits in 90909 exceeding the radix of the system.

6. (1). First,

$$2ax+x^2 = (b-a-x)^2 = b^2+(a+x)^2-2b\,(a+x),$$

$$x = \frac{b-2a}{2-}.$$

(2). First

$$(a+x)^{\frac{2}{3}} - 5\,(a^2-x^2)^{\frac{1}{3}} = -4\,(a-x)^{\frac{2}{3}}.$$

Divide by $(a-x)^{\frac{2}{3}}$; then

$$\left(\frac{a+x}{a-x}\right)^{\frac{2}{3}} - 5\left(\frac{a+x}{a-x}\right)^{\frac{1}{3}} = -4,$$

$$\therefore \left(\frac{a+x}{a-x}\right)^{\frac{1}{3}} = \pm\tfrac{3}{2}+\tfrac{5}{2} = 4 \text{ and } 1,$$

$$\therefore a+x = 64\,(a-x) \text{ and } a-x,$$

$$\therefore x = \frac{63a}{65} \text{ and } x = 0.$$

(3). First,

$$\sqrt{x}+\sqrt{18}+\frac{\sqrt{(x-18)}}{\sqrt{x}} = \sqrt{x}-\frac{\sqrt{x}}{\sqrt{(x-18)}},$$

$$\therefore \sqrt{(18x)}+\sqrt{(x-18)} = -\frac{x}{\sqrt{(x-18)}},$$

$$\therefore \sqrt{\{18x\,(x-18)\}}+x-18 = -x,$$

$$\therefore 18x\,(x-18) = 4x^2-72x+18^2,$$

$$\therefore x^2-18x = \frac{162}{7},$$

$$\therefore x = 9 \pm \sqrt{\frac{729}{7}} = 9 \pm \frac{27}{\sqrt{7}}.$$

(4). Cubing the first, and substituting

$$216 = 126+18x^{\frac{1}{4}}y^{\frac{1}{5}},$$

$$\therefore 4x^{\frac{1}{4}}y^{\frac{1}{5}} = 20,$$

$$\therefore x^{\frac{1}{4}} - y^{\frac{1}{5}} = \pm 4,$$

$$\therefore x^{\frac{1}{4}} = 5 \text{ and } 1,$$

$$y^{\frac{1}{4}} = 1 \text{ and } 5,$$

$$\therefore \left.\begin{array}{l} x = 625 \text{ and } 1 \\ y = 1 \text{ and } 3125 \end{array}\right\}.$$

(5). From the first,

$$(x+y)\,(x^2+y^2) = 13,$$

and the second is misprinted (in the original paper) for

$$x^2 y^2 (x^2+y^2) = 468.$$

Make $x+y = u$, $xy = v$; then

$$x^2+y^2 = u^2-2v,$$

and substituting, we get

$$\left.\begin{aligned} u\ (u^2-2v) &= 13 \\ v^2\ (u^2-2v) &= 468 \end{aligned}\right\},$$

$$\therefore\ \frac{v^2}{u} = 36,$$

$$\text{and } u = \frac{v^2}{36}. \ldots . (1),$$

$$\therefore\ v^2\left(\frac{v^4}{1296} - 2v\right) = 468. \ldots (2),$$

which, being in the form of a quadratic, will give v^3, and then v; then, by (1), we shall have u; thence $x+y$ and xy, which will give x and y. Two pairs of simultaneous values of x and y are

$$\left.\begin{aligned} x = -2,\ y = 3 \\ y = -2,\ x = 3 \end{aligned}\right\}.$$

7. See *Private Tutor*, vol. i. p. 23.

$$(a^2-x^2)^{\frac{2}{3}} = a^{\frac{1}{3}}\left(1 - \frac{x^2}{a^2}\right)^{\frac{2}{3}},$$

$$= a^{\frac{1}{3}}\left(1 - \frac{2}{3}.\frac{x^2}{a^2} - \frac{1}{9}.\frac{x^4}{a^4} - \frac{4}{81}.\frac{x^6}{a^6} - \frac{7}{243}.\frac{x^8}{a^8} - \&\text{c.}\right).$$

Also, $\{xy-\sqrt{(3yz)}\}^{\frac{5}{4}} = (xy)^{\frac{5}{4}}\left\{1 - \left(\frac{3z}{x^2y}\right)^{\frac{1}{2}}\right\}^{\frac{5}{4}}$,

$$= (xy)^{\frac{5}{4}}\left\{1 - \frac{5}{4}.\left(\frac{3z}{x^2y}\right)^{\frac{1}{2}} - \frac{15}{32}.\frac{z}{x^2y} - \frac{15}{128}.\frac{z}{x^2y}\sqrt{\frac{3z}{x^2y}} - \frac{35}{2048}.\frac{z^2}{x^2y^2} - \&\text{c.}\right\}.$$

8. $\sqrt{(\sqrt{54}-\sqrt{30})} = \sqrt{\dfrac{\sqrt{54}+\sqrt{24}}{2}} - \sqrt{\dfrac{\sqrt{54}-\sqrt{24}}{2}}$

$$= \sqrt{\frac{5\sqrt{6}}{2}} - \sqrt{\frac{\sqrt{6}}{2}} = \tfrac{1}{2}\sqrt[4]{600} - \sqrt[4]{24}.$$

$\sqrt[3]{(26-15\sqrt{3})} = 2-\sqrt{3}$. (*Wood*, art. 259.)

9. The series is Geometric, the common ratio being $\frac{y^2}{x^2}$;

$$\therefore \text{S} = (x-y).\frac{\frac{y^{2n}}{x^{2n}} - 1}{\frac{y}{x} - 1} = \frac{x^{2n}-y^{2n}}{x^{2n-1}},$$

$$\Sigma = \frac{x-y}{1-\frac{y}{x}} = x,$$

$$\therefore \text{S} : \Sigma :: x^{2n}-y^{2n} : x^{2n}.$$

10. The enunciation of this question is ambiguous. "*Different throws*" may be understood both as not producing the same *aggregate of the numbers shown by the dice,* and also as giving faces of the dice not all the same in any two throws. We shall adopt the latter signification.

Now, each of the faces marked 1, 2, 3, 4, 5, 6, 7, 8, of the octahedron, may appear with each of those of the dodecahedron, thus producing 8×12 combinations.

Also, each face of the hexahedron, or common die, marked 1, 2, 3, 4, 5, 6, may appear with each of the preceding 96 combinations; so that the three dice produce 96×6 combinations.

Lastly, each face of the tetrahedron may appear with each of these combinations, producing in all $4 \times 6 \times 96$, or 2304 different throws.

11. See pp. 42, 46, and 190.

12. $(a+bx+cx^2+\&c.\ldots)^n = a^n + n.(bx+cx^2+dx^3+ex^4+\ldots\ldots)$

$$+\frac{n(n-1)}{1.\,2}(bx+cx^2+dx^3+\&c.\,.)^2$$

$$+\frac{n(n-1)(n-2)}{1.\,2.\,3}(bx+cx^2+\ldots.)^3$$

$$+\frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}(bx+\ldots\ldots)^4+\&c.$$

$\therefore$ the coefficient of x^4 is

$$ne + \frac{n(n-1)}{1.\ 2}(c^2+2bd) + \frac{n(n-1)(n-2)}{1.\ 2}.\ b^2 c$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}.\ b^4.$$

CATHARINE HALL, 1828.

1. $\frac{2}{5}$ of $9\frac{1}{6} = \frac{2}{5} \times \frac{55}{6} = \frac{11}{3}$,
   $6\frac{3}{2}$ of $\frac{1}{25}$ of $\frac{1}{6} = \frac{15}{2} \times \frac{1}{25} \times \frac{1}{6} = \frac{1}{20} = .05$.

2. Ans. $283l.\ 11s.\ 11d\frac{1}{2}\frac{1}{8}$.

3. Interest

$$= nr\,\mathrm{P} = 2\tfrac{1}{2} \times \frac{4\frac{1}{2}}{100} \times 284\tfrac{1}{2} = \frac{5121}{160} = 32l.\ 0s.\ 1\tfrac{1}{2}d.$$

4. It $= 3s.\ 4d.$

5. Sum $= \dfrac{a^2-ab+b^2}{a^2-b^2}$.

Difference $= \dfrac{ab}{a-b}$,

Product $= a^3-b^3$,

Quotient $= a^{n-1}+a^{n-2}b+a^{n-3}b^2+ \ldots\ldots b^{n-1}$.

6. (1). Ans. $x = 7$.
   (2). Ans. $x = a$ and b.
   (3). First,

$$x^2-2x+5+6\sqrt{(x^2-2x+5)} = 16,$$
$$\therefore\ \sqrt{(x^2+2x+5)} = -3 \pm 5 = 2 \text{ and} -8,$$
$$\therefore\ x^2-2x = -1 \text{ and } 59,$$
$$\therefore\ x = 1 \text{ and } \pm \sqrt{60}.$$

   (4). First, $x-1 = \dfrac{2}{\sqrt{x}}(\sqrt{x}+1)$,

$$\therefore\ (\sqrt{x}+1)(x-\sqrt{x}-2) = 0,$$
$$\therefore\ \sqrt{x} = -1,$$
$$\text{and } x-\sqrt{x} = 2,$$

$$\therefore \sqrt{x} = \tfrac{1}{2} \pm \tfrac{3}{2} = 2 \text{ and } -1,$$
$$\therefore x = 4 \text{ and } 1.$$

(5). See p. 240.

7. See *Wood*, art. 90.

8. The lowest fractions are

$$\frac{c+d}{f+2x} \text{ and } \frac{x-1}{x+2}.$$

9. $(a-b)^{-\frac{1}{2}} = a^{-\frac{1}{2}}\left(1-\frac{b}{a}\right)^{-\frac{1}{2}},$

$$= a^{-\frac{1}{2}}\left(1+\tfrac{1}{2}\frac{b}{a} + \frac{3}{8}\cdot\frac{b^2}{a^2} + \&\text{c.}\right)$$
$$= \frac{1}{\sqrt{a}} + \frac{1}{2}\cdot\frac{b}{a\sqrt{a}} + \frac{3}{8}\cdot\frac{b^2}{a^2\sqrt{a}} + \&\text{c.}$$

10. See p. 218.

11. See p. 200.

12. *Wood*, art. 402.

13. See *Private Tutor*, Alg. part i. p. 291.

$$\sqrt{15625} = 125,$$
$$\sqrt{(1+x)} = 1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\&\text{c.}$$

14. $\sqrt{(\sqrt{-3}-1)} = \sqrt{\frac{2-1}{2}} + \sqrt{\frac{2+1}{2}}\cdot\sqrt{-1},$

$$= \sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{3}{2}}\cdot\sqrt{-1},$$
$$\sqrt[3]{(10+\sqrt{108})} = 1+\sqrt{3}.$$

15. *Wood*, art. 452.

16. Let x be the number of crowns; y that of the guineas; then

$$5x + 21y = 20 \times 20 = 400,$$

$$\left.\begin{array}{l} y = 0,\ 5,\ 10,\ 15,\ 20\ \ldots\ldots \\ x = 80,\ 59,\ 38,\ 17, -2\ \ldots \end{array}\right\}\text{Ans. 4 ways.}$$

JESUS COLLEGE, 1828.

1.
$$\begin{array}{r|l}
5 = \frac{1}{4} & 530 \\ \hline
2.\,6 = \frac{1}{2} & 132\ .\ 10 \\
 & \ \ 66\ .\ \ 5 \\ \hline
 & 198l.\ 15s.\ 0d.
\end{array}$$

2. *Euclid*, Book i. Prop. 35.

3. The product is

$$pa^{\frac{3}{4}}x^{\frac{23}{12}} - a^{\frac{7}{3}}x^{\frac{11}{12}} - a^{\frac{10}{3}}x^{-\frac{1}{12}} - (p-2)\frac{a^{\frac{13}{3}}}{x}.$$

4. First to the straight line apply a parallelogram $= 2 \times$ the given figure; half of this parallelogram will be a $\triangle$ on the given base, which is the $\triangle$ required, excepting the vertical angle.

Upon the base describe a segment of a circle containing an angle = the given angle of the vertex. This segment will be met by the upper side of the parallelogram in two points P, P′. Join P or P′ with the extremities of the base, and there will result the $\triangle$ required.

5. $$\frac{5}{13\,.\,4} = \frac{1}{4} \div \frac{2}{3} = \frac{3}{8} = .375.$$

6. (1). $$x = \tfrac{2}{3},$$

(2). $$x = \tfrac{1}{4},\ y = \tfrac{1}{5},$$

(3). First $x^2 + \dfrac{64}{x^2} = \dfrac{10}{3} \cdot \left(x + \dfrac{8}{x}\right)$,

$$\therefore \left(x + \frac{8}{x}\right)^2 - \frac{10}{3}\left(x + \frac{8}{x}\right) = 16,$$

$$\therefore x + \frac{8}{x} - \frac{5}{3} = \pm\sqrt{\left(\frac{25}{9} + 16\right)} = \pm\sqrt{\frac{169}{9}},$$

$$\therefore x + \frac{8}{x} = \frac{5 \pm 13}{3} = 6, \text{ and } -\frac{8}{3},$$

$$\therefore\ x^2-6x = -8, \text{ and } x^2+\tfrac{8}{3}x = -8,$$

$$\therefore\ x = 3 \pm 1, \text{ and } x = -\frac{4}{3} \pm \frac{\sqrt{-56}}{3},$$

$$\text{or, } x = 4, 2 \text{ and } \frac{-4 \pm \sqrt{-56}}{2}.$$

7. Euc.

8. Let $x-y$, x, $x+y$ be the numbers; then

$$3x = 15,\ (x^2-y^2)x = 120,$$
$$\therefore\ x = 5,\ y = \pm 1.$$

9. Similar figures $\propto$ squares of the homologous sides; $\therefore$ if A, B, C be the areas of the figures on the sides a, b, c, respectively, c being opposite the right angle, then we have

$$\begin{aligned} &\mathrm{A} : \mathrm{B} &&:: a^2 : b^2, \\ \therefore\ &\mathrm{A} : \mathrm{A}+\mathrm{B} &&:: a^2 : a^2+b^2, \\ \text{and }\ &\mathrm{C} : \mathrm{A} &&:: c^2 : a^2, \\ \therefore\ &\mathrm{C} : \mathrm{A}+\mathrm{B} &&:: c^2 : a^2+b^2 :: 1 : 1. \end{aligned}$$
(Euc. 47. b. i.).

10. *Wood*, art. 189.

11. *Woodhouse's* Trig.

12. *Woodhouse's* Trig.

13. The root is

$$\frac{x^2}{2} + ax + \frac{a^2}{3}.$$

14. If α be the area of the $\triangle$, A, B, C the angles, and a, b, c the opposite sides; then

$$a^2 = 2\alpha.\ \frac{\sin \mathrm{A}}{\sin \mathrm{B}.\ \sin \mathrm{C}},$$

$$b^2 = 2\alpha.\ \frac{\sin \mathrm{B}}{\sin \mathrm{A}.\ \sin \mathrm{C}},$$

$$c^2 = 2\alpha.\ \frac{\sin \mathrm{C}}{\sin \mathrm{A}.\ \sin \mathrm{B}}.$$

15. $$\begin{aligned} \text{Log } .225 &= \log 225-3, \\ &= \log 25+2\log 3-3, \\ &= 2-2\log 2+2\log 3-3, \end{aligned}$$

$$= 2\,(\log 3 - \log 2) - 1,$$
$$= 2 \times .176091 - 1 = \bar{1}.352182.$$

16. $\dfrac{9\,x^3 - 72}{3\,x - 6} = 3.\,\dfrac{x^3 - 8}{x - 2} = 3.\,(x^2 + 2\,x + 4).$

17. Euc. B. xi.

18. Since $\cos P + \cos Q = 2\cos\dfrac{P+Q}{2}\cos\dfrac{P-Q}{2}$,

$\therefore\ \cos(60 + A) + \cos(60 - A) = 2\cos 60.\ \cos A.$
$$= 2 \times \tfrac{1}{2}\cos A = \cos A.$$

19. Ans. $\frac{20}{12} - \frac{21}{14} = \frac{5}{3} - \frac{3}{2} = \frac{1}{6}$ shilling, or 2 pence.

20. Let x be the common ratio; then
$$\tfrac{1}{2}\,x^4 = 128\,;\ \therefore\ x = 4,$$
and the series is
$$\tfrac{1}{2},\ 2,\ 8,\ 32,\ 128.$$

MAGDALENE COLLEGE, 1828.

1. $\frac{4}{5}$ of $4d. = \frac{16}{5}$ of a penny $= \frac{16}{25}$ of five-pence.

2. The value $= .878125l. = 17s.\ 6\frac{3}{4}d.$

3. Number of days $\propto \dfrac{\text{length} \times \text{width} \times \text{depth}}{\text{men} \times \text{hours}}$,

$\therefore$ if x be the number of days required, we have
$$x : \frac{64.27.18}{32.12} :: 16 : \frac{72.18.12}{36.8},$$
$$\therefore\ x = \frac{64.27.18}{24} \times \frac{8.36}{72.18.12} = \frac{8.9.18}{2.9.3} = 24.$$

4. For the proof, see *Wood.*

(1). The G. C. M. = 127, and the fraction $= \dfrac{131}{191}$.

(2). The G. C. M. $= x - 2$ . . . . $= \dfrac{x^2}{x-2}$.

(3). The G. C. M. $= y - 1$ . . . . $= \dfrac{2y^2 + 3\,y - 5}{7\,y + 5}$.

5. The quotient is

$$a^m + 3a^{m-1}b^n - 6a^{m-2}b^{2m}.$$

6. That $a^{-m} = \frac{1}{a^m}$ is a definition; not a theorem.

See *Wood*, art. 53.

$$\left(a^{-\frac{m}{n}}\right)^{-\frac{p}{q}} = \left(\frac{1}{a^{\frac{m}{n}}}\right)^{-\frac{p}{q}} = a^{\frac{mp}{nq}}.$$

7. The square root $= 3y^2 + qy - \frac{py}{2}$.

The cube root $= 258$.

8. *Wood*, art. 189.

9. (1). Ans. $x = \frac{204}{251}$.

(2). Ans. $x = \frac{d}{c}$.

(3). From the first,

$$4x + y = 17;$$

from the second,

$$2x - 3y = -9,$$
$$x = 3,\ y = 5.$$

(4). Ans. $x = 5$ and $\frac{27}{5}$.

(5). From the first equation,

$$\frac{x^2}{y^2} + \frac{2x}{\sqrt{y}} + y + \frac{x}{y} + \sqrt{y} = 20,$$

$$\text{or } \left(\frac{x}{y} + \sqrt{y}\right)^2 + \left(\frac{x}{y} + \sqrt{y}\right) = 20,$$

$$\therefore \frac{x}{y} + \sqrt{y} = -\tfrac{1}{2} \pm \tfrac{9}{2} = 4 \text{ and } -5 \quad \ldots\ldots \quad (1).$$

$$\therefore \frac{4y-8}{y} + \sqrt{y} = 4 \text{ and } -5,$$

$$\therefore -\frac{8}{y} + \sqrt{y} = 0,$$

$$\therefore y^{\frac{3}{2}} = 8 \text{ and } y = 4,$$

$$\therefore\ x = 4y - 8 = 8.$$

If the -5 be taken, we get

$$y^{\frac{3}{2}} + 9y = 8,$$

$$\therefore\ y^{\frac{3}{2}} + 1 = -9.\ (y-1),$$

$$\therefore \text{ dividing by } \sqrt{y} + 1,$$

$$y - \sqrt{y} + 1 = -9\,(\sqrt{y} - 1),$$

$$\therefore\ y + 8\sqrt{y} = 8,$$

$$\text{and } \sqrt{y} = -4 \pm \sqrt{24},$$

$$\text{whence } y \text{ and } \therefore\ x.$$

10. Let $\frac{a}{x}$, a, ax be the numbers; then

$$\left.\begin{aligned} a\left(\frac{1}{x} + 1 + x\right) &= 13 \\ \frac{1 + \frac{1}{x}}{1+x} &= \tfrac{1}{3} \end{aligned}\right\}$$

$\therefore$ the numbers are

$$1,\ 3,\ 9.$$

11. Let x be number to whom he gave 9 pence; then

$$9x + 15\,(n-x) = 12p,$$

$$\therefore\ x = \frac{15n - 12p}{6} = \frac{5n-4p}{2},$$

12. $\left(\frac{a+x}{a}\right)^m = \left(1 + \frac{x}{a}\right)^m = 1 + m\frac{x}{a}$ nearly, if x be small.

13. *Private Tutor*, vol i. p. 22 gives the proof.

$$(\sqrt{a} + \sqrt{b})^4 = (\sqrt{a})^4 + 4\,(\sqrt{a})^3\sqrt{b} + 6\,(\sqrt{a})^2(\sqrt{b})^2$$
$$+ 4\sqrt{a}.(\sqrt{b})^3 + b^2,$$
$$= a^2 + 4a\sqrt{(ab)} + 6ab + 4b\sqrt{(ab)} + b^2.$$

14. $(1+1)^n = 1 + n + \frac{n\,(n-1)}{1.\ 2} + \frac{n\,(n-1)\,(n-2)}{1.\ 2.\ 3} + \&c.$

15. $\sqrt{(7 + 6\sqrt{-2})} = 3 + \sqrt{-2}$,

See *Wood*, art. 258.

16. The Present Value $= \dfrac{M}{1+nr}$ (*Wood*, art. 392),

$$= \frac{725\frac{4}{5}}{1+\frac{1}{2}\cdot\dfrac{3\frac{1}{2}}{100}} = 713l.\ 6s.\ 4d.\ \frac{28}{407},$$

Whence discount $=$ future value $-$ present, is found.

17. The present value of an Annuity A, for p years, is (*Wood*, art. 402.)

$$P = \frac{1-\dfrac{1}{R^p}}{R-1}\cdot A,$$

and for $p+q$ years it is

$$P' = \frac{1-\dfrac{1}{R^{p+q}}}{R-1}\ A,$$

$\therefore$ the present value of A, to commence after p years and continue q years, is

$$P'-P = \frac{A}{R^p}\cdot\frac{1-\dfrac{1}{R^q}}{R-1}.$$

Hence, if V be the value of the Estate for n years, and V′ that of the reversion,

$$V = \frac{1-\dfrac{1}{R^n}}{R-1}\cdot P,\quad \frac{1}{V'} = \frac{P}{R^n}\cdot\ \frac{1}{R-1},$$

$$\therefore V : V' :: 1-\frac{1}{R^n} : \frac{1}{R^n},$$

$$:: R^n - 1 : 1,$$

$$\text{in which } R = 1 + \frac{r}{100}.$$

18. Log $720 = \log 10 + \log 9 + \log 8$,

$= 1 + 2\log 3 + 3\log 2 = \&c.$

$\log 7.2 = \log 72 - \log 10 = 2\log 3 + 3\log 2 - 1 = \&c.$

19. First $(a^2-b^2)^{2x-2} = \dfrac{(a-b)^{2x}}{(a+b)^2}$

$$\therefore \frac{(a^2-b^2)^{2x}}{(a^2-b^2)^2} = \frac{(a-b)^{2x}}{(a+b)^2},$$

$$\therefore (a+b)^{2x} = (a+b)^2,$$

$$\therefore x = 1.$$

20. Let N be the number, d_0, d_1, &c. the digits, then

$$\begin{aligned} N &= d_0 + d_1 r + d_2 r^2 + d_3 r^3 + \dots \\ &= d_0 + d_1(r+1-1) + d_2(r+1-1)^2 + \&c. \\ &= d_0 - d_1 + d_2 - d_3 + \dots \\ &\qquad + (r+1) \times Q, \end{aligned}$$

whence the proposition is manifest.

21. See *Barlow's* Theory of Numbers on this subject.

$$\begin{array}{r} 19 \;.\; 6 \\ 14 \;.\; 7 \\ \hline 273 \;.\; 0 \\ 11 \;.\; 4 \;.\; 6 \\ \hline 284 \;.\; 4 \;.\; 6 \\ \hline \end{array}$$

Ans. 284 square feet, and 54 square inches.

TRINITY HALL, 1828.

1. $\frac{2}{3}$ of $\frac{6}{11}$ of half-a-crown $= \frac{2}{3} \times \frac{6}{11} \times \frac{5}{2}$ shillings,

$= \frac{10}{11}$ shillings,

$\therefore$ fraction required $= \frac{10}{231}$.

2. $3.\ 11\frac{3}{4} = 3 + \frac{11}{12} + \frac{3}{48} = 3\frac{47}{48} = \frac{191}{48}$ shillings,

$\therefore$ decimal required $= \frac{191}{960} = .1997\dots$

3. Ans. 169*l*. 12*s*. 7*d*. $\frac{17}{28}$.

4. Interest $= nr\text{P} = 9 \times \frac{9}{200} \times 999$,

$= 404l.\ 11s.\ 10d.\ \frac{3}{4}.\frac{1}{5}$.

5. Horses $\propto \dfrac{\text{pounds}}{\text{days}}$,

$\therefore$ if x be the number required,

$$x : \tfrac{28}{7} :: 7 : \tfrac{14}{20}.$$

$$\therefore x = 40.$$

6. *Wood*, art. 76.

7. (1). Product $= x^2 - 4$.

(2). . . . . $= x^5 - 11x^4 + 38x^3 - 45x^2 + 27x - 10$.

(3). . . . . $= x^4 - \dfrac{77}{60}x^3 + \dfrac{71}{120}x^2 - \dfrac{x}{15} + \dfrac{1}{120}$.

(4). . . . . $= a^3 - b^2$.

8. (1). Quotient $= x + a$.

(2). . . . . . $= x^2 + 5x + 6$ (for 24 write -24).

(3). . . . . . $= x^4 + \frac{1}{3}x^3 + \frac{1}{9}x^2 + \frac{1}{27}x + \frac{1}{81}$.

(4). . . . . . $= \left(2\sqrt[4]{x} + \dfrac{y}{2}\right)\left(4\sqrt{x} + \dfrac{y^2}{4}\right)$.

(5). . . . . . $= a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

9. *Wood*, art. 93.

10. Sum $= \dfrac{2}{a-b}$.

11. For proof, see *Wood*.

$$\text{Quotient} = \frac{x+a}{x-b} \times \frac{x-a}{x+b} = \frac{x^2-a^2}{x^2-b^2}.$$

12. *Wood*, art. 110, &c.

13. $\sqrt{5.76} = 2.4$; $\sqrt{0.576} = .758$, &c.

$$\sqrt[3]{\left(\frac{a^3}{8} - \frac{3a^2b^{\frac{1}{2}}}{2} + 6ab - 8b^{\frac{3}{2}}\right)} = \frac{a}{2} - 2b^{\frac{1}{2}}.$$

14. (1). Ans. $x = 12$.

(2). Ans. $x = 25$.

(3). Ans. $\left.\begin{array}{l} x = 3 \text{ and } 2 \\ y = 2 \text{ and } 3 \end{array}\right\}$.

(4). Ans. $x = \frac{7 \pm \sqrt{-56}}{3}$.

(5). Dividing by $\sqrt[5]{(1-x)^2}$

we get

$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{5}} - \left(\frac{1+x}{1-x}\right)^{\frac{1}{5}} = -1,$$

$$\therefore \left(\frac{1+x}{1-x}\right)^{\frac{1}{5}} = \tfrac{1}{2} \pm \sqrt{(\tfrac{1}{4}-1)},$$

$$= \tfrac{1}{2} \pm \frac{\sqrt{-3}}{2},$$

$$\therefore \frac{1+x}{1-x}$$

$$= \frac{1 \pm 5\sqrt{-3} + 10(-3) \pm 10.(\sqrt{-3})^3 + 5(\sqrt{-3})^4 \pm (\sqrt{-3})^5}{32}$$

$$= \frac{16 \mp 16\sqrt{-3}}{32} = \frac{1 \mp \sqrt{-3}}{2},$$

$$\therefore 2 + 2x = 1 \mp \sqrt{-3} - x \pm \sqrt{-3}.\, x,$$

$$\therefore x(3 \mp \sqrt{-3}) = -1 \mp \sqrt{-3},$$

$$\therefore x = \frac{-1 \mp \sqrt{-3}}{3 \mp \sqrt{-3}} = \frac{(-1 \mp \sqrt{-3})(3 \pm \sqrt{-3})}{12},$$

$$= \pm \frac{\sqrt{-1}}{\sqrt{3}}.$$

15. *Wood*, art. 162.

16. *Wood*, art. 189.

17. Sum $= (4-9)\,5 = -25$,

$$\text{Sum} = \tfrac{3}{2}.\, \frac{1}{1+\frac{2}{3}} = \tfrac{9}{10}.$$

18. The Common Difference is evidently 2, $\therefore$ the series is
$-5, -3, -1, 1, 3, 5, 7, 9, 11$, &c.

19. $s = a.\, \dfrac{r^n - 1}{r-1}$,

$$\therefore r^n = 1 + (r-1)\frac{s}{n},$$

$$\therefore n \log r = \log \left\{1 + (r-1)\frac{s}{n}\right\}, \text{ \&c.}$$

20. See *Private Tutor*, vol. i. p. 21, &c.

21. Cubing the first

$$a+b\sqrt{a} = m^3+3\,m^2n\sqrt{a}+3\,mn^2a+n^3a\sqrt{a},$$

and since the surd terms cannot be destroyed by the rational terms,

$$\begin{aligned} a &= m^3+3\,mn^2a,\\ b\sqrt{a} &= 3\,m^2n\sqrt{a}+n^3a\sqrt{a},\\ \therefore\ a-b\sqrt{a} &= m^3-3\,m^2n\sqrt{a}+3\,mn^2a-n^3a\sqrt{a},\\ &= (m-n\sqrt{a})^3,\\ \therefore\ &\text{\&c.} \end{aligned}$$

22. $\sqrt{(6+2\sqrt{5})} = \sqrt{\dfrac{6+4}{2}}+\sqrt{\dfrac{6-4}{2}} = \sqrt{5}+1,$

$$\sqrt[3]{(2+\sqrt{5})} = \tfrac{1}{2}.\,(1+\sqrt{5})$$

23. See *Private Tutor*, vol. i. p. 26.

24. See *Barlow's* Theory of Numbers.

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1. Ans. 317*l.* 0*s.* $2\frac{3}{4}$*d.*

2. (1). $\dfrac{19}{190000} = \dfrac{1}{10000} = .0001.$

(2). $\dfrac{8160000}{4} = 2040000.$

(3). The quotient $= a^2-\frac{4}{3}\,a+\frac{3}{4}.$

(4). $\dfrac{a^3-b}{a^{\frac{1}{3}}-b^{\frac{1}{6}}} = \dfrac{(a^{\frac{1}{3}})^6-(b^{\frac{1}{6}})^6}{a^{\frac{1}{3}}-b^{\frac{1}{6}}} = \dfrac{(a^{\frac{1}{3}})^3-(b^{\frac{1}{6}})^3}{a^{\frac{1}{3}}-b^{\frac{1}{6}}}.\,(a+b^{\frac{1}{2}})$

$$= (a^{\frac{2}{3}}+a^{\frac{1}{3}}\,b^{\frac{1}{6}}+b^{\frac{1}{3}})\,(a+b^{\frac{1}{2}}).$$

3. $\sqrt{17} = 4.1231056.$

4. *Wood*, art. 78, 109, *et seq.*

5. *Wood*, art. 92.

6. $\dfrac{a^2+b^2 \pm (a-b)^2}{a^2-b^2} = \dfrac{2\,(a^2+b^2-a\,b)}{a^2-b^2}$, or $\dfrac{2\,ab}{a^2-b^2}$.

7. (1). $x+12 = \dfrac{1}{16}$, $\therefore\ x = -\dfrac{191}{12}$.

(2). $x = 6$ and 1.

(3). From the second, it is found that $xy = 8$; whence

$$x-y = \pm 7,$$
$$\text{and } x-y = 9,$$
$$\therefore\ x = 8 \text{ and } 1,$$
$$y = 1 \text{ and } 8.$$

(4). First,
$$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= 1 \\ \frac{1}{x} + \frac{1}{z} &= \frac{1}{2} \\ \frac{1}{y} + \frac{1}{z} &= \frac{1}{3} \end{aligned}\right\}.$$

Adding them together,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}\cdot\left(1 + \frac{1}{2} + \frac{1}{3}\right),$$
$$= \frac{11}{12},$$
$$\therefore\ \frac{1}{z} = -\frac{1}{12},\ \frac{1}{y} = \frac{5}{12},\ \frac{1}{x} = \frac{7}{12},$$
$$\therefore\ x = \frac{12}{7},\ y = \frac{12}{5},\ z = -12.$$

8. *Wood*, art. 182.

9. *Wood*, art. 222.

(1). Sum $= (4-10\times\frac{1}{2})\,\frac{11}{2} = -\,\frac{11}{2}$.

(2). Sum to n terms $= 4.\ \dfrac{(\frac{3}{4})^n-1}{\frac{3}{4}-1} = \dfrac{4^{10}-3^{10}}{4^8}$,

which is easily computed.

Sum to $\infty = 4.\ \dfrac{1}{1-\frac{3}{4}} = 16$.

(3). See p. 218.

10. See p. 240. No. 8. The required means are $\frac{12}{5}$ and 3.

11. Discount = future — present value $= 153 - \dfrac{M}{1+nr}$,

(*Wood*, art. 392),

$$= 153 - \frac{153}{1+\frac{r}{2}} = 3, \text{ by the question,}$$

$$\therefore \frac{153\,r}{2} = 3 + \frac{3r}{2},$$

$$150\,r = 6,$$

$$\therefore r = \frac{1}{25} = \frac{4}{100}, \text{ or 4 per cent.}$$

12. See p. 254. No. 14.

13. The whole number $= n + \dfrac{n\,(n-1)}{1.\ 2} + \ldots$

But $(1+1)^n = 1+n+\dfrac{n\,(n-1)}{1.\ 2} + \&c.$

&c.

14. See *Private Tutor*, vol. i. p. 22.

The fourth term of $(a+b)^n$ is

$$\frac{n\,(n-1)\,(n-2)}{1.\ 2.\ 3}\,a^{n-3}\,b^3,$$

and $\therefore$ that required is

$$\frac{\frac{2}{3}.\,(\frac{2}{3}-1)\,(\frac{2}{3}-2)}{1.\ 2.\ 3}\left(\frac{\sqrt{x}}{2}\right)^{\frac{2}{3}-3}.\left(\frac{\sqrt[3]{y}}{3}\right)^3,$$

$$\text{or } \frac{2\sqrt[3]{2}}{2187}.\,\frac{y}{x^{\frac{7}{6}}}.$$

15. $\sqrt{(10\sqrt{7}+22)} = 1+\sqrt{7}$.

16. See *Private Tutor*, vol. i. p. 327.

17. The possible parts cannot equal impossible parts, being essentially distinct by definition.

18. $N = .\alpha\beta\gamma\alpha\beta\gamma\ldots\ldots$

$$1000\,N = \alpha\beta\gamma.\alpha\beta\gamma\ldots\ldots = \alpha\beta\gamma + N,$$

$$\therefore\ N = \frac{\alpha\beta\gamma}{999} = \frac{\gamma + 10\beta + 10^2\alpha}{999}.$$

19.
$$\begin{array}{r}
6541)\,1433221\,6\,(1456 \\
\underline{6541} \\
44612 \\
\underline{36124} \\
54551 \\
\underline{45665} \\
55536 \\
\underline{55536} \\
\ldots\ldots
\end{array}$$

20. See *Barlow's* Theory of Numbers.

21. $\sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \&c.}}$

22. There is no difficulty in this.

23. See p. 209. No. 16.

FINIS.

ERRATA.

P. 43 *last line for* $\frac{4s.\ 5d.}{5} = \frac{53}{9}$ *read* $\frac{4s.\ 9d.}{5} = \frac{19}{20}$.

— 44 *first line for* $\frac{53}{60}$ *read* $\frac{19}{20}$, and *for* $\frac{53}{150}$ *read* $\frac{19}{50}$.

— 44 *line* 5 *for* 16874 *read* 16875.

— 47 *line* 17 *for* 10×2 *read* 10×3, and *for* 132 *read* 187.

— 53 *line* 21 *for* 48 *read* 44, and 22 *for* $\frac{24}{7} = 3\frac{3}{7}$ *read* $\frac{68}{21} = 3\frac{5}{21}$.

— 54 *line* 19 *for* 38 *read* 38 and $-$ 30.

— 56 *line* 11 *for* $\frac{115}{124}$ *read* $\frac{115}{104}$.

— 57 *line* 6 *for* $x^n + x^{n-1}y$ + &c. *read* $x^{n-1} + x^{n-2}y$ + &c.
19 *for* x^{2n-2} *read* x^{2n-1}.

— 58 *line* 17 *for* 12 y *read* 84 y.

$$= \frac{166}{11} = 15\frac{1}{11} \text{ read } = 2.$$

line 18 *for* $19\frac{5}{7} + \frac{1}{11} = 19\frac{62}{17}$ *read* 5.

— 61 *lines* 30 and 31 *for* 1. 471213 *read* 1. 4771213.

— 63 *line* 11 *for* If *read* It.

— 64 *line* 10 *for* 12 *read* 6, and *for* $-$ 21 *read* $-\frac{21}{2}$.

— 65 *line* 5 *for* 487 *read* 507, and 7 *for* $1\frac{7}{48}$ *read* $\frac{47}{48}$.

— 68. Question 24 is conducted on wrong principles.

The probability required is $\frac{tt'\,l''}{tt'l'' + ll'l''} = \frac{3.\ 4.\ 1}{3.\ 4.\ 1 + 1.\ 1.\ 6} = \frac{2}{3}$.

See *Laeroix's* Probab. p. 249.

— 103 *line* 13 refer to p. 79.

— 240 *line* 5 *for* bequadratic *read* biquadratic.

Errata in the Questions *of Euclid, Arithmetic and Algebra.*

— 37 Question 10 α *for* $-ab$ in denom. *read* $+ab$.

— 42 - - - - 8 (2) *for* $x-3$ - - - - - - *read* $\frac{x-3}{9}$.

— 42 - - - - 8 (4) *for* x^2+2x - - - - *read* x^2-2x.

— 42 - - - - 10 (2) *for* $d-b$ - - - - *read* $a-b$.

— 45 - - - - 6 *for* x^2-y^{2n} - - - - *read* $x^{2n}-y^{2n}$.

— 50 - - - - 2 *for* least - - - - *read* last.

— 86 - - - - 9 (3). The series should be

$$\frac{1}{1+\sqrt{x}} + \frac{1}{(1-x)\,(1+\sqrt{x})} + \frac{1}{(1-x)^2.\,(1+\sqrt{x})} + \text{\&c.}$$

— 93 - - - - 17. This is erroneous in the original paper.

— 97 - - - - 11 (2) *for* $= \sqrt{\frac{x+y}{3x}}$ *read* $- \sqrt{\frac{x+y}{3x}}$.

This a misprint in the original paper.

— 110 - - - - 1 *for* $12p+p$ - - - - - *read* $12p+q$.

— 112 - - - - 3 *for* $x^4-x^2y-x^2y+y^3$ - - - - *read* $x^4-x^2y-x^2y^2+y^3$.

— 118 - - - - 6 (5) *for* x^2y - - - - - *read* x^2y^2.

This error is in the original paper.